Constructing special Lagrangian m-folds in $\mathbb{C}^m$ by evolving quadrics

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1 Introduction

This is the second in a series of papers constructing explicit examples of special Lagrangian submanifolds (SL m-folds) in $\mathbb{C}^m$. The first paper of the series [7] studied SL m-folds with large symmetry groups, and subsequent papers [8, 9, 10, 11] construct examples of SL 3-folds in $\mathbb{C}^3$ using evolution equations, symmetries, ruled submanifolds and integrable systems.

The principal motivation for these papers is to lay the foundations for a study of the singularities of compact special Lagrangian m-folds in Calabi–Yau m-folds, particularly in low dimensions such as $m = 3$. Special Lagrangian m-folds in $\mathbb{C}^m$, and especially *special Lagrangian cones*, should provide local models for singularities of SL m-folds in Calabi–Yau m-folds.

Understanding such singularities will be essential in making rigorous the explanation of Mirror Symmetry of Calabi–Yau 3-folds $X, \hat{X}$ proposed by Strominger, Yau and Zaslow [13], which involves dual 'fibrations' of $X, \hat{X}$ by special Lagrangian 3-tori, with some singular fibres. It will also be important in resolving conjectures made by the author [6], which attempt to define an invariant of Calabi–Yau 3-folds by counting special Lagrangian homology 3-spheres.

The paper falls into three parts. The first, this section and §2, is introductory. The second part, §3–§4, describes a general construction of special Lagrangian m-folds N in $\mathbb{C}^m$, depending on a set of *evolution data* (P, χ), where P is an $(m-1)$-submanifold in $\mathbb{R}^n$. Then N is the subset of $\mathbb{C}^m$ swept out by the image of P under a 1-parameter family of linear or affine maps $\phi_t : \mathbb{R}^n \to \mathbb{C}^m$, which satisfy a first-order, nonlinear o.d.e. in t.

Examples of sets of evolution data will be given in §4, together with some progress towards a classification of such data. The simplest interesting sets of evolution data occur when $n = m$ and P is a nondegenerate quadric in $\mathbb{R}^m$. In the third part, §5–§7, we apply the construction to these examples.

In this case $\phi_t(\mathbb{R}^m)$ must be a Lagrangian plane in $\mathbb{C}^m$ for each t. Thus N is fibred by quadrics in Lagrangian planes $\mathbb{R}^m$ in $\mathbb{C}^m$. The construction of §3–§4 will also be used in the sequel to this paper [8], with different evolution data, to construct families of SL 3-folds in $\mathbb{C}^3$.

The construction has both a linear and an affine version. In the linear version

we begin with a centred quadric Q in $\mathbb{R}^m$, such as an ellipsoid or a hyperboloid, and evolve its image under linear maps $\phi_t : \mathbb{R}^m \to \mathbb{C}^m$. This will be studied in §5 for $\mathbb{C}^m$, and in more detail when $m = 3$ in §6. In the affine version we begin with a non-centred quadric Q in $\mathbb{R}^m$, such as a paraboloid, and evolve its image under affine maps $\phi_t : \mathbb{R}^m \to \mathbb{C}^m$. This will be studied in §7.

In some cases the family $\{\phi_t : t \in \mathbb{R}\}$ turns out to be *periodic* in t. The corresponding SL m-folds in $\mathbb{C}^m$ are then closed, and are interesting as local models for singular behaviour of SL m-folds in Calabi–Yau m-folds. Section 5.5 studies the periodicity conditions, and proves our main result, Theorem 5.9, on the existence of large families of SL m-folds in $\mathbb{C}^m$ with interesting topology, including cones on $\mathcal{S}^a \times \mathcal{S}^b \times \mathcal{S}^1$ for $a+b = m-2$. When $m = 3$ this gives many new examples of SL T^2-cones in $\mathbb{C}^3$, which are discussed in §6.

In contrast to the manifolds of [7], the SL m-folds N in $\mathbb{C}^m$ that we construct generically have only finite symmetry groups. However, we shall show in §4.3 that every set of evolution data (P, χ) actually admits a large symmetry group G, which is locally transitive on P. This 'internal symmetry group' does act on N, but not by automorphisms of $\mathbb{C}^m$. So we can think of the construction as embodying a symmetry assumption, but not of the most obvious kind.

Some of the SL m-folds we construct (those in §5 from evolving ellipsoids) are already known, having been found by Lawlor [12] and completed by Harvey [4, p. 139–143]. But as far as the author knows, the other examples are new. The SL T^2-cones in $\mathbb{C}^3$ are related to integrable systems results on harmonic tori in $\mathbb{CP}^m$. We discuss the connection in §6.2.

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2 Special Lagrangian submanifolds in $\mathbb{C}^m$

We begin by defining *calibrations* and *calibrated submanifolds*, following Harvey and Lawson [5].

Definition 2.1 Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane* V on M is a vector subspace V of some tangent space T_xM to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k-plane on M then $g|_V$ is a Euclidean metric on V, so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V, which is a k-form on V.

Now let φ be a closed k-form on M. We say that φ is a *calibration* on M if for every oriented k-plane V on M we have $\varphi|_V \leqslant \mathrm{vol}_V$. Here $\varphi|_V = \alpha \cdot \mathrm{vol}_V$ for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leqslant \mathrm{vol}_V$ if $\alpha \leqslant 1$. Let N be an oriented submanifold of M with dimension k. Then each tangent space T_xN for $x \in N$ is an oriented tangent k-plane. We say that N is a *calibrated submanifold* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

It is easy to show that calibrated submanifolds are automatically *minimal*

submanifolds [5, Th. II.4.2]. Here is the definition of special Lagrangian submanifolds in $\mathbb{C}^m$, taken from [5, §III].

Definition 2.2 Let $\mathbb{C}^m$ have complex coordinates $(z_1, \dots, z_m)$, and define a metric g, a real 2-form ω and a complex m-form Ω on $\mathbb{C}^m$ by

$$g = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m),$$
$$\text{and} \quad \Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m.$$

Then $\operatorname{Re}\Omega$ and $\operatorname{Im}\Omega$ are real m-forms on $\mathbb{C}^m$. Let L be an oriented real submanifold of $\mathbb{C}^m$ of real dimension m, and let $\theta \in [0, 2\pi)$. We say that L is a *special Lagrangian submanifold* of $\mathbb{C}^m$ if L is calibrated with respect to $\operatorname{Re}\Omega$, in the sense of Definition 2.1. We will often abbreviate 'special Lagrangian' by 'SL', and 'm-dimensional submanifold' by 'm-fold', so that we shall talk about SL m-folds in $\mathbb{C}^m$.

As in [6, 7] there is also a more general definition of special Lagrangian submanifolds involving a *phase* $\mathrm{e}^{i\theta}$, but we will not use it in this paper. Harvey and Lawson [5, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds.

Proposition 2.3 *Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. Then L admits an orientation making it into an SL submanifold of $\mathbb{C}^m$ if and only if $\omega|_L \equiv 0$ and $\operatorname{Im}\Omega|_L \equiv 0$.*

Note that an m-dimensional submanifold L in $\mathbb{C}^m$ is called *Lagrangian* if $\omega|_L \equiv 0$. Thus special Lagrangian submanifolds are Lagrangian submanifolds satisfying the extra condition that $\operatorname{Im}\Omega|_L \equiv 0$, which is how they get their name.

3 SL m-folds from evolution equations

The construction of special Lagrangian m-folds we shall study in this paper is based on the following theorem, which was proved in [7, Th. 3.3].

Theorem 3.1 *Let P be a compact, orientable, real analytic $(m-1)$-manifold, χ a real analytic, nonvanishing section of $\Lambda^{m-1}TP$, and $\phi : P \to \mathbb{C}^m$ a real analytic embedding (immersion) such that $\phi^*(\omega) \equiv 0$ on P. Then there exists $\epsilon > 0$ and a unique family $\{\phi_t : t \in (-\epsilon, \epsilon)\}$ of real analytic maps $\phi_t : P \to \mathbb{C}^m$ with $\phi_0 = \phi$, satisfying the equation*

$$\left(\frac{\mathrm{d}\phi_t}{\mathrm{d}t}\right)^b = (\phi_t)_*(\chi)^{a_1\ldots a_{m-1}}(\operatorname{Re}\Omega)_{a_1\ldots a_{m-1}a_m} g^{a_m b}, \tag{1}$$

using the index notation for (real) tensors on $\mathbb{C}^m$. Define $\Phi : (-\epsilon, \epsilon) \times P \to \mathbb{C}^m$ by $\Phi(t, p) = \phi_t(p)$. Then $N = \operatorname{Im}\Phi$ is a nonsingular embedded (immersed) special Lagrangian submanifold of $\mathbb{C}^m$.

The proof relies on a result of Harvey and Lawson [5, Th. III.5.5], which says that if P is a real analytic $(m-1)$-submanifold of $\mathbb{C}^m$ with $\omega|_P \equiv 0$, then there is a locally unique SL submanifold N containing P. They assume P is real analytic as their proof uses Cartan–Kähler theory, which works only in the real analytic category. But this is no loss, as by [5, Th. III.2.7] all nonsingular SL m-folds in $\mathbb{C}^m$ are real analytic.

We interpret equation (1) as an *evolution equation* for (compact) real analytic $(m-1)$-submanifolds $\phi(P)$ of $\mathbb{C}^m$ with $\omega|_{\phi(P)} \equiv 0$, and think of the variable t as time. The theorem says that given such a submanifold $\phi(P)$, there is a 1-parameter family of diffeomorphic submanifolds $\phi_t(P)$ satisfying a first-order o.d.e., with $\phi_0(P) = \phi(P)$, that sweep out an SL m-fold in $\mathbb{C}^m$.

The condition that P be compact is not always necessary in Theorem 3.1. Whether P is compact or not, in a small neighbourhood of any $p \in P$ the maps ϕ_t always exist for $t \in (-\epsilon, \epsilon)$ and some $\epsilon > 0$, which may depend on p. If P is compact we can choose an $\epsilon > 0$ valid for all p, but if P is noncompact there may not exist such an ϵ. If P is not compact but we know for other reasons that there exists a family $\{\phi_t : t \in (-\epsilon, \epsilon)\}$ satisfying (1) and $\phi_0 = \phi$, then the conclusions of the theorem still hold.

Now Theorem 3.1 should be thought of as an *infinite-dimensional* evolution problem, since the evolution takes place in an infinite-dimensional family of real analytic $(m-1)$-submanifolds. This makes the o.d.e. difficult to solve explicitly, so that the theorem, in its current form, is unsuitable for constructing explicit SL m-folds. However, there is a method to reduce it to a *finite-dimensional* evolution problem.

Suppose we can find a special class $\mathcal{C}$ of real analytic $(m-1)$-submanifolds P of $\mathbb{C}^m$ with $\omega|_P \equiv 0$, depending on finitely many real parameters $c_1, \ldots, c_n$, such that the evolution equation (1) stays within the class $\mathcal{C}$. Then (1) reduces to a first order o.d.e. on $c_1, \ldots, c_n$, as functions of t. Thus we have reduced the infinite-dimensional problem of evolving submanifolds in $\mathbb{C}^m$ to a finite-dimensional o.d.e., which we may be able to solve explicitly.

This method was used in [7], where $\mathcal{C}$ was a set of $(m-1)$-dimensional group orbits. We now present a more advanced construction based on the same idea, in which $\mathcal{C}$ consists of the images of an $(m-1)$-submanifold P in $\mathbb{R}^n$ under linear or affine maps $\mathbb{R}^n \to \mathbb{C}^m$. We describe the linear case first.

Definition 3.2 Let $2 \leqslant m \leqslant n$ be integers. A set of *linear evolution data* is a pair (P, χ), where P is an $(m-1)$-dimensional submanifold of $\mathbb{R}^n$, and $\chi : \mathbb{R}^n \to \Lambda^{m-1}\mathbb{R}^n$ is a linear map, such that $\chi(p)$ is a nonzero element of $\Lambda^{m-1}TP$ in $\Lambda^{m-1}\mathbb{R}^n$ for each nonsingular point $p \in P$. We suppose also that P is not contained in any proper vector subspace $\mathbb{R}^k$ of $\mathbb{R}^n$.

Let $\mathrm{Hom}(\mathbb{R}^n, \mathbb{C}^m)$ be the real vector space of linear maps $\phi : \mathbb{R}^n \to \mathbb{C}^m$, and define $\mathcal{C}_P$ to be the subset of $\phi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{C}^m)$ such that

(i) $\phi^*(\omega)|_P \equiv 0$, and

(ii) $\phi|_{T_pP} : T_pP \to \mathbb{C}^m$ is injective for all p in a dense open subset of P.

If $\phi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{C}^m)$ then (i) holds if and only if $\phi^*(\omega) \in V_P$, where V_P is the vector subspace of elements of $\Lambda^2(\mathbb{R}^n)^*$ which restrict to zero on P. This is a quadratic condition on ϕ. Also (ii) is an open condition on ϕ. Thus $\mathcal{C}_P$ is an open set in the intersection of a finite number of quadrics in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{C}^m)$. Let $\mathbb{R}^m$ be a Lagrangian plane in $\mathbb{C}^m$. Then any linear map $\phi : \mathbb{R}^n \to \mathbb{R}^m$ satisfies (i), and generic linear maps $\phi : \mathbb{R}^n \to \mathbb{R}^m$ satisfy (ii). Hence $\mathcal{C}_P$ is nonempty.

Note that the requirement that χ be both linear in $\mathbb{R}^n$ and tangent to P at every point is a very strong condition on P and χ. Thus sets of linear evolution data are quite rigid things, and not that easy to construct. We will give some examples in §4. First we show how to construct SL m-folds in $\mathbb{C}^m$ using linear evolution data.

Theorem 3.3 *Let (P, χ) be a set of linear evolution data, and use the notation above. Suppose $\phi \in \mathcal{C}_P$. Then there exists $\epsilon > 0$ and a unique real analytic family $\{\phi_t : t \in (-\epsilon, \epsilon)\}$ in $\mathcal{C}_P$ with $\phi_0 = \phi$, satisfying the equation*

$$\left(\frac{\mathrm{d}\phi_t}{\mathrm{d}t}(x)\right)^b = (\phi_t)_*(\chi(x))^{a_1 \ldots a_{m-1}} (\mathrm{Re}\,\Omega)_{a_1 \ldots a_{m-1} a_m} g^{a_m b} \tag{2}$$

for all $x \in \mathbb{R}^n$, using the index notation for tensors in $\mathbb{C}^m$. Furthermore, $N = \{\phi_t(p) : t \in (-\epsilon, \epsilon),\ p \in P\}$ is a special Lagrangian submanifold in $\mathbb{C}^m$ wherever it is nonsingular.

Before we prove the theorem, here are some remarks about it. Equation (2) is a first-order o.d.e. upon ϕ_t, and should be compared with equation (1) of Theorem 3.1. The key point to note is that as χ is linear, the right hand side of (2) is linear in x, and so (2) makes sense as an evolution equation for linear maps ϕ_t. However, the right hand side of (2) is a homogeneous polynomial of order $m-1$ in ϕ_t, so for $m > 2$ it is a *nonlinear* o.d.e.

Also observe that (2) works for ϕ_t in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{C}^m)$, and not just $\mathcal{C}_P$. If the evolution starts in $\mathcal{C}_P$, then it stays in $\mathcal{C}_P$ for small t. But it can be helpful to think of the evolution as happening in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{C}^m)$ rather than in $\mathcal{C}_P$, because $\mathcal{C}_P$ may be singular, but $\mathrm{Hom}(\mathbb{R}^n, \mathbb{C}^m)$ is nonsingular. Thus, we do not run into problems when the evolution hits a singular point of $\mathcal{C}_P$.

Proof of Theorem 3.3. As above, equation (2) is a well-defined, first-order o.d.e. upon ϕ_t in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{C}^m)$ of the form $\frac{\mathrm{d}\phi_t}{\mathrm{d}t} = Q(\phi_t)$, where $Q : \mathrm{Hom}(\mathbb{R}^n, \mathbb{C}^m) \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{C}^m)$ is a homogeneous polynomial of degree $m-1$. The existence for some $\epsilon > 0$ of a unique, real analytic solution $\{\phi_t : t \in (-\epsilon, \epsilon)\}$ in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{C}^m)$ with initial value $\phi_0 = \phi$ follows easily from standard results on ordinary differential equations.

The rest of the proof follows that of Theorem 3.1, given in [7, Th. 3.3], with small modifications. The compactness of P in Theorem 3.1 was used only to prove existence of the family $\{\phi_t : t \in (-\epsilon, \epsilon)\}$, which we have already established, so we don't need to suppose P is compact. The evolution equation (1) in Theorem 3.1 is exactly the restriction of (2) from $\mathbb{R}^n$ to P. Thus the

proof in Theorem 3.1 that N is special Lagrangian also applies here, wherever N is nonsingular.

It remains only to show that $\{\phi_t : t \in (-\epsilon, \epsilon)\}$ lies in $\mathcal{C}_P$, rather than just in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{C}^m)$. Now $\omega|_N \equiv 0$ as N is special Lagrangian, and this implies that $\phi_t^*(\omega)|_P \equiv 0$ for $t \in (-\epsilon, \epsilon)$. So part (i) of Definition 3.2 holds for ϕ_t. But part (ii) is an open condition, and it holds for $\phi_0 = \phi$ as $\phi \in \mathcal{C}_P$. Thus, making $\epsilon > 0$ smaller if necessary, we see that $\phi_t \in \mathcal{C}_P$ for all $t \in (-\epsilon, \epsilon)$. $\square$

Next we generalize the ideas above from linear to *affine* (linear+constant) maps ϕ. Here are the analogues of Definition 3.2 and Theorem 3.3.

Definition 3.4 Let $2 \leqslant m \leqslant n$ be integers. A set of *affine evolution data* is a pair (P, χ), where P is an $(m-1)$-dimensional submanifold of $\mathbb{R}^n$, and $\chi : \mathbb{R}^n \to \Lambda^{m-1}\mathbb{R}^n$ is an affine map, such that $\chi(p)$ is a nonzero element of $\Lambda^{m-1}TP$ in $\Lambda^{m-1}\mathbb{R}^n$ for each nonsingular $p \in P$. We suppose also that P is not contained in any proper affine subspace $\mathbb{R}^k$ of $\mathbb{R}^n$.

Let $\mathrm{Aff}(\mathbb{R}^n, \mathbb{C}^m)$ be the affine space of affine maps $\phi : \mathbb{R}^n \to \mathbb{C}^m$, and define $\mathcal{C}_P$ to be the subset of $\phi \in \mathrm{Aff}(\mathbb{R}^n, \mathbb{C}^m)$ satisfying parts (i) and (ii) of Definition 3.2. Then $\mathcal{C}_P$ is nonempty, and is an open set in the intersection of a finite number of quadrics in $\mathrm{Aff}(\mathbb{R}^n, \mathbb{C}^m)$.

Theorem 3.5 *Let (P, χ) be a set of affine evolution data, and use the notation above. Suppose $\phi \in \mathcal{C}_P$. Then there exists $\epsilon > 0$ and a unique real analytic family $\{\phi_t : t \in (-\epsilon, \epsilon)\}$ in $\mathcal{C}_P$ with $\phi_0 = \phi$, satisfying* (2) *for all $x \in \mathbb{R}^n$, using the index notation for tensors in $\mathbb{C}^m$. Furthermore, $N = \{\phi_t(p) : t \in (-\epsilon, \epsilon),\ p \in P\}$ is a special Lagrangian submanifold in $\mathbb{C}^m$ wherever it is nonsingular.*

Now the affine case in $\mathbb{R}^n$ can in fact be reduced to the linear case in $\mathbb{R}^{n+1}$, by regarding $\mathbb{R}^n$ as the hyperplane $\mathbb{R}^n \times \{1\}$ in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. Then any affine map $\phi : \mathbb{R}^n \to \mathbb{C}^m$ extends to a unique linear map $\phi' : \mathbb{R}^{n+1} \to \mathbb{C}^m$. Thus Theorem 3.5 follows immediately from Theorem 3.3.

4 Examples of evolution data

We now give examples of sets of linear and affine evolution data (P, χ), in order to apply the construction of §3. We begin in §4.1 by showing that quadrics in $\mathbb{R}^m$ are examples of evolution data with $m = n$. The corresponding SL m-folds will be studied in §5–§7.

Section 4.2 gives two trivial examples of evolution data, and classifies sets of evolution data in the cases $m = 2$ and $m = n$. Then §4.3 considers the *symmetries* of sets of evolution data, and shows that every set of evolution data (P, χ) has a large symmetry group G which acts locally transitively on P. Finally §4.4 discusses the classification of evolution data, and the rôle of the symmetry group.

4.1 Quadrics in $\mathbb{R}^m$ as examples of evolution data

A large class of examples of evolution data arise as *quadrics* in $\mathbb{R}^m$, with $n = m$.

Theorem 4.1 *Let $\mathbb{R}^m$ have coordinates $(x_1, \dots, x_m)$, and for $j = 1, \dots, m$ define $e_j \in \mathbb{R}^m$ by $x_j = 1$ and $x_k = 0$ for $j \neq k$. Let $Q : \mathbb{R}^m \to \mathbb{R}$ be a quadratic polynomial. Define $\chi : \mathbb{R}^m \to \Lambda^{m-1}\mathbb{R}^m$ by*

$$\begin{aligned}\chi(x) &= \mathrm{d}Q(x) \cdot (e_1 \wedge \cdots \wedge e_m) \\ &= \sum_{j=1}^{m} (-1)^{j-1} \frac{\partial Q(x)}{\partial x_j} e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_m.\end{aligned} \tag{3}$$

Let P be the quadric $\{x \in \mathbb{R}^n : Q(x) = c\}$ for some $c \in \mathbb{R}$, and suppose P is nonempty and nondegenerate.

If Q is a homogeneous quadratic polynomial then (P, χ) is a set of linear evolution data in the sense of Definition 3.2 with $n = m$, and otherwise (P, χ) is a set of affine evolution data in the sense of Definition 3.4 with $n = m$.

The proof of this theorem is simple. As Q is quadratic, $\mathrm{d}Q$ is linear or affine, so $\chi(x)$ is linear or affine in x. Since $\chi = \mathrm{d}Q \cdot (e_1 \wedge \cdots \wedge e_m)$ and P is a level set of Q, it is clear that χ lies in $\Lambda^{m-1}TP$ on P. We leave the details to the reader.

Here are three examples in $\mathbb{R}^m$, using notation as above.

Example 4.2 Let $1 \leqslant a \leqslant m$, and define P and χ by

$$\begin{aligned}P &= \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - \cdots - x_m^2 = 1\}, \\ \chi &= 2 \sum_{j=1}^{a} (-1)^{j-1} x_j \, e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_m \\ &\quad - 2 \sum_{j=a+1}^{m} (-1)^{j-1} x_j \, e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_m.\end{aligned}$$

Then P is nonsingular in $\mathbb{R}^m$, and (P, χ) is a set of linear evolution data.

Example 4.3 Let $m/2 \leqslant a < m$, and define P and χ by

$$\begin{aligned}P &= \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - \cdots - x_m^2 = 0\}, \\ \chi &= 2 \sum_{j=1}^{a} (-1)^{j-1} x_j \, e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_m \\ &\quad - 2 \sum_{j=a+1}^{m} (-1)^{j-1} x_j \, e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_m.\end{aligned}$$

Then P is a quadric cone in $\mathbb{R}^m$ with an isolated singular point at 0, and (P, χ) is a set of linear evolution data.

Example 4.4 Let $(m-1)/2 \leqslant a \leqslant m-1$, and define P and χ by

$$P = \left\{(x_1,\ldots,x_m) \in \mathbb{R}^m : x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - \cdots - x_{m-1}^2 + 2x_m = 0\right\},$$

$$\chi = 2(-1)^{m-1} e_1 \wedge \cdots \wedge e_{m-1} + 2\sum_{j=1}^{a} (-1)^{j-1} x_j e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_m$$
$$- 2 \sum_{j=a+1}^{m-1} (-1)^{j-1} x_j \, e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_m.$$

Then P is nonsingular in $\mathbb{R}^m$, and (P,χ) is a set of affine evolution data.

The classifications of centred quadrics in $\mathbb{R}^m$ up to linear automorphisms, and of general quadrics in $\mathbb{R}^m$ up to affine automorphisms, are well known. Our construction is unchanged under linear or affine automorphisms of $\mathbb{R}^m$. It can be shown that all interesting sets of evolution data arising from Theorem 4.1 are isomorphic to one of the cases of Examples 4.2–4.4, under an affine automorphism of $\mathbb{R}^m$ and a rescaling of χ.

Here we exclude quadrics admitting a translational symmetry group $\mathbb{R}^k$ for $k \geqslant 1$ as uninteresting, since they lead to special Lagrangian submanifolds N in $\mathbb{C}^m$ with the same translational symmetry group. It then follows that N is a product $N' \times \mathbb{R}^k$ in $\mathbb{C}^{m-k} \times \mathbb{C}^k$, where N' is special Lagrangian in $\mathbb{C}^{m-k}$. Degenerate quadrics with dimension less than $m-1$ are also excluded.

4.2 Two trivial constructions of evolution data

Next we consider evolution data not arising from the quadric construction above. The following two examples are rather trivial constructions of evolution data, which do not yield interesting SL m-folds in $\mathbb{C}^m$.

Example 4.5 Let $n \geqslant 2$, choose any nonzero linear or affine map $\chi : \mathbb{R}^n \to \mathbb{R}^n$, and let P be any integral curve of χ, regarded as a vector field in $\mathbb{R}^n$. Then (P,χ) is a set of linear or affine evolution data with $m = 2$. Furthermore, every set of evolution data with $m = 2$ comes from this construction.

Thus, using the method of §3, one can construct many examples of special Lagrangian 2-folds in $\mathbb{C}^2$. But special Lagrangian 2-folds in $\mathbb{C}^2$ are equivalent to holomorphic curves with respect to an alternative complex structure, and so are anyway very easy to construct.

Example 4.6 Let (P,χ) be a set of evolution data in $\mathbb{R}^n$, with P an $(m-1)$-manifold, and let $k \geqslant 1$. Write $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$, with coordinates $(x_1,\ldots,x_n, x_{n+1},\ldots,x_{n+k})$. Define

$$P' = P \times \mathbb{R}^k \quad \text{and} \quad \chi' = \chi \wedge \tfrac{\partial}{\partial x_{n+1}} \wedge \cdots \wedge \tfrac{\partial}{\partial x_{n+k}}.$$

Then (P',χ') is a set of evolution data in $\mathbb{R}^{n+k}$, with P' an $(m+k-1)$-manifold. All SL $(m+k)$-folds N' in $\mathbb{C}^{m+k}$ constructed using (P',χ') split as products $N \times \mathbb{R}^k$ in $\mathbb{C}^m \times \mathbb{C}^k$, where N is an SL m-fold in $\mathbb{C}^m$ constructed using (P,χ).

Combining these two examples we can make (uninteresting) examples of evolution data for any m, n with $2 \leqslant m \leqslant n$. In particular, when $n = m$ we have:

Example 4.7 Let $a, \ldots, f \in \mathbb{R}$ be not all zero, and let γ be an integral curve of the vector field $(ax_1 + bx_2 + e)\frac{\partial}{\partial x_1} + (cx_1 + dx_2 + f)\frac{\partial}{\partial x_2}$ in $\mathbb{R}^2$. Let $m \geqslant 2$, write $\mathbb{R}^m = \mathbb{R}^2 \times \mathbb{R}^{m-2}$, and define $P = \gamma \times \mathbb{R}^{m-2}$ and

$$\chi = (ax_1+bx_2+e)\tfrac{\partial}{\partial x_1} \wedge \tfrac{\partial}{\partial x_3} \wedge \cdots \wedge \tfrac{\partial}{\partial x_m} + (cx_1+dx_2+f)\tfrac{\partial}{\partial x_2} \wedge \cdots \wedge \tfrac{\partial}{\partial x_m}.$$

Then (P, χ) is a set of evolution data with $n = m$. If $e = f = 0$ then it is linear, and otherwise affine.

These and the examples of §4.1 exhaust the examples with $m = n$.

Proposition 4.8 *Every set of linear or affine evolution data with $m = n$ is isomorphic either to one of the quadric examples of §4.1, or to one constructed in Example 4.7.*

Proof. Let (P, χ) be a set of linear or affine evolution data in $V = \mathbb{R}^m$, with $m = n$. Let α be a nonzero element of $\Lambda^m V^*$, and define $\beta : V \to V^*$ by $\beta = \alpha \cdot \chi$, where '$\cdot$' is the natural product $\Lambda^m V^* \times \Lambda^{m-1} V \to V^*$. Then β is a linear or affine 1-form on V.

The zeros of β form a distribution $\mathcal{D}$ of hyperplanes in V wherever β is nonzero. The *curvature* of $\mathcal{D}$ is $(\mathrm{d}\beta)|_{\mathcal{D}}$. Now clearly $\beta|_P \equiv 0$, since χ is nonzero and tangent to P at each point of P, so β is nonzero along P and $\mathcal{D}|_P = TP$. Therefore P is an *integral submanifold* of $\mathcal{D}$, so the curvature of $\mathcal{D}$ vanishes along P.

This shows that $\mathrm{d}\beta|_P \equiv 0$. Clearly, this is equivalent to $\beta \wedge \mathrm{d}\beta$ being zero along P, that is, zero in $\Lambda^3 V^*$ rather than restricted to P. But β is linear or affine and $\mathrm{d}\beta$ is constant, so $\beta \wedge \mathrm{d}\beta$ is linear or affine. As P is not contained in any proper linear or affine subspace of V (as appropriate), we see that $\beta \wedge \mathrm{d}\beta$ is zero on all of V.

There are now two possibilities:

(a) $\mathrm{d}\beta = 0$, or

(b) $\mathrm{d}\beta = \gamma \wedge \delta$ for linearly independent $\gamma, \delta \in V^*$, and $\beta \in \langle \gamma, \delta \rangle_{\mathbb{R}}$ at each point in V.

This is because if $\mathrm{d}\beta$ is nonzero and not of the form $\gamma \wedge \delta$, then $\beta \wedge \mathrm{d}\beta = 0$ if and only if $\beta = 0$, but we know β is nonzero on P.

In case (a), we can write $\beta = \mathrm{d}Q$ for $Q : V \to \mathbb{R}$ a quadratic polynomial, which is homogeneous if β is linear. Then Q is constant along P (assuming P connected), so P is a subset of $P' = \{v \in V : Q(v) = c\}$. Thus, case (a) is one of the quadric examples of §4.1. In case (b), we choose coordinates $(x_1, \ldots, x_m)$ on V with $\gamma = \mathrm{d}x_1$ and $\delta = \mathrm{d}x_2$, and it is then easy to show that we are in the situation of Example 4.7. $\square$

4.3 Symmetry groups of evolution data

We shall now show that every set of evolution data (P, χ) has a symmetry group G which is locally transitive on P. For simplicity we work in the linear case; the corresponding result for affine evolution data may easily be obtained by replacing linear by affine actions.

Theorem 4.9 *Let (P, χ) be a set of linear evolution data, with P a connected, nonsingular $(m-1)$-submanifold in $\mathbb{R}^n$. Then there exists a connected Lie subgroup G in $\mathrm{GL}(n, \mathbb{R})$ with Lie algebra $\mathfrak{g}$, such that P is an open set in a G-orbit in $\mathbb{R}^n$, and χ is G-invariant. Furthermore, there is a natural, surjective, G-equivariant linear map $L : \Lambda^{m-2}(\mathbb{R}^n)^* \to \mathfrak{g}$.*

Proof. Define a linear map $L : \Lambda^{m-1}(\mathbb{R}^n)^* \to \mathfrak{gl}(n, \mathbb{R})$ by $L(\alpha) = \chi \cdot \alpha$, where we regard χ as an element of $(\mathbb{R}^n)^* \otimes \Lambda^{m-1}\mathbb{R}^n$, and '$\cdot$' is the natural contraction $\Lambda^{m-1}\mathbb{R}^n \times \Lambda^{m-2}(\mathbb{R}^n)^* \to \mathbb{R}^n$, so that $\chi \cdot \alpha \in (\mathbb{R}^n)^* \otimes \mathbb{R}^n = \mathfrak{gl}(n, \mathbb{R})$. Let $\mathfrak{g}$ be the Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ generated by $\operatorname{Im} L$, so that L maps $\Lambda^{m-2}(\mathbb{R}^n)^* \to \mathfrak{g}$. Let G be the unique connected Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ with Lie algebra $\mathfrak{g}$.

Regard elements of $\mathfrak{gl}(n, \mathbb{R})$ as linear vector fields on $\mathbb{R}^n$. Then at each $p \in P \subset \mathbb{R}^n$ we have $L(\alpha)|_p = \chi|_p \cdot \alpha$. Since $\chi|_p \in \Lambda^{m-1}T_pP$ by definition, we see that $L(\alpha)|_p \in T_pP$. So the vector fields $L(\alpha)$ are tangent to P. But the Lie bracket of two vector fields tangent to P is also tangent to P. Hence, as $\mathfrak{g}$ is generated from $\operatorname{Im} L$ by the Lie bracket, every vector field in $\mathfrak{g}$ is tangent to P.

Since P is nonsingular, we have $\chi|_p \neq 0$ for all $p \in P$, by definition. Thus the map $\Lambda^{m-2}(\mathbb{R}^n)^* \to T_pP$ given by $\alpha \mapsto L(\alpha)|_p$ is *surjective*. So the vector fields in $\mathfrak{g}$ span T_pP for all $p \in P$. Therefore the action of the Lie algebra $\mathfrak{g}$ on P is *locally transitive*. It follows that P is locally isomorphic to an orbit of G in $\mathbb{R}^n$, and as P is connected, it must be an open set in a G-orbit.

Next we prove that χ is G-invariant, which is not quite as obvious as it looks. Let $1 \leqslant i_1 < \cdots < i_{m-2} \leqslant n$, set $\alpha = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{m-2}}$, and define $v = L(\alpha)$. We shall show that $\mathcal{L}_v\chi = 0$, where $\mathcal{L}_v$ is the Lie derivative. First observe that v is a linear combination of terms $x_i \frac{\partial}{\partial x_j}$ with $j \neq i_k$ for $k = 1, \ldots, m-2$. It follows easily that $\mathcal{L}_v\alpha = 0$. But then

$$0 = \mathcal{L}_v v = \mathcal{L}_v(\chi \cdot \alpha) = (\mathcal{L}_v\chi) \cdot \alpha + \chi \cdot (\mathcal{L}_v\alpha) = (\mathcal{L}_v\chi) \cdot \alpha. \tag{4}$$

Now $\chi|_P$ is a nonvanishing section of $\Lambda^{m-1}TP$ and v is tangent to P, we see that $\mathcal{L}_v\chi|_P = \lambda\chi|_P$ for some smooth function $\lambda : P \to \mathbb{R}$. As $(\mathcal{L}_v\chi) \cdot \alpha = 0$ by (4), restricting to P gives $\lambda\chi \cdot \alpha = 0$ on P, that is, $\lambda v \equiv 0$ on P. Therefore $\lambda \equiv 0$ or $v \equiv 0$ on P. But if $v \equiv 0$ then clearly $\lambda \equiv 0$. Thus $\mathcal{L}_v\chi \equiv 0$ on P.

Since P lies in no proper vector subspace of $\mathbb{R}^n$, and $\mathcal{L}_v\chi$ is linear, this implies that $\mathcal{L}_v\chi \equiv 0$. This holds whenever $v = L(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{m-2}})$ for $1 \leqslant i_1 < \cdots < i_{m-2} \leqslant n$. Such forms are a basis for $\Lambda^{m-2}(\mathbb{R}^n)^*$. So $\mathcal{L}_v\chi = 0$ for all $v \in \operatorname{Im} L$, and therefore for all $v \in \mathfrak{g}$. As G is connected, this shows that χ is G-invariant.

It remains to show that $L : \Lambda^{m-1}(\mathbb{R}^n)^* \to \mathfrak{g}$ is G-equivariant and surjective. The G-equivariance is now obvious, as χ is G-invariant. So $\operatorname{Im} L$ is a G-invariant

subspace of $\mathfrak{g}$, that is, an *ideal* in $\mathfrak{g}$. But then $\operatorname{Im} L$ is closed under the Lie bracket. As $\operatorname{Im} L$ generates $\mathfrak{g}$ we have $\mathfrak{g} = \operatorname{Im} L$, and L is surjective. $\square$

As an example, consider the linear evolution data (P, χ) given in Examples 4.2 and 4.3. In both cases G is the identity component of $\mathrm{SO}(a, m-a)$. In Example 4.2, each connected component of P is an orbit of G. In Example 4.3, P is singular at 0, and each component of $P \setminus \{0\}$ is an orbit of G.

Now fix $m = 3$. Then L maps $(\mathbb{R}^n)^* \to \mathfrak{g}$. It can be shown that either

(a) P is contained in no affine hyperplane in $\mathbb{R}^n$, and $\operatorname{Ker} L = 0$; or

(b) There exists a nonzero linear map $f : \mathbb{R}^n \to \mathbb{R}$ such that P is contained in the affine hyperplane $f \equiv 1$ in $\mathbb{R}^n$, and $\operatorname{Ker} L = \langle \mathrm{d}f \rangle_{\mathbb{R}}$.

In case (a), L is an isomorphism, so that $\mathbb{R}^n \cong \mathfrak{g}^*$. Thus, P is an open set in G-orbit in the coadjoint representation $\mathfrak{g}^*$ of G, that is, P is locally a *coadjoint orbit*. In case (b) we will see in [8, §4] that $(\mathbb{R}^n)^*$ is also a Lie algebra, an extension of $\mathfrak{g}$ by $\mathbb{R}$, and P is again a coadjoint orbit. Note that in case (b) (P, χ) reduces to a set of affine evolution data in $\mathbb{R}^{n-1}$.

In the sequel to this paper [8], we will use these ideas to construct a correspondence between sets of evolution data with $m = 3$, and symplectic 2-manifolds with a transitive, Hamiltonian symmetry group. This will enable us to write down several interesting sets of evolution data with $m = 3$ and $n > 3$, and study the corresponding families of SL 3-folds in $\mathbb{C}^3$.

4.4 Discussion

Let us survey what we know about of sets of evolution data so far. Evolution data depends on two integers m, n with $2 \leqslant m \leqslant n$. In §4.2 we classified all sets of evolution data with $m = 2$ and $m = n$, and constructed some not very interesting examples for any m, n with $2 \leqslant m \leqslant n$. The ideas of [8, §4] will give us a good picture of the set of all evolution data with $m = 3$, and could probably be developed into a classification without great difficulty.

What we lack at present is an understanding of sets of evolution data with $3 < m < n$. We can state this as:

Problem. Find and classify examples of sets of evolution data with $3 < m < n$, which do not arise from lower-dimensional examples via the product construction of Example 4.6.

Theorem 4.9 suggests a possible method of constructing examples. One should start with a likely-looking connected Lie group G and a representation V of G, and find the $(m-1)$-dimensional orbits $\mathcal{O}$ of G in V, and then look for G-invariant elements χ of $V^* \otimes \Lambda^{m-1} V$ which are tangent to $\mathcal{O}$. Note that if χ is nonzero and tangent to $\mathcal{O}$ at one point, then it is at every point.

The case $n = m + 1$ may also be tractable by a more direct approach. For instance, affine evolution data with $n = m$, which we understand, can be interpreted as linear evolution data with $n = m + 1$.

Next we discuss the geometric meaning of Theorem 4.9. It shows that any set of evolution data (P, χ) in $\mathbb{R}^n$ has a symmetry group G, acting on $\mathbb{R}^n$ in a locally transitive way on P and preserving χ. Take P to be a G-orbit in $\mathbb{R}^n$, so that G acts globally on P, rather than just locally. Let us ask, how is G related to the special Lagrangian m-folds N in $\mathbb{C}^m$ constructed from (P, χ) in §3?

As N is naturally isomorphic to $P \times (-\epsilon, \epsilon)$ or $P \times \mathbb{R}$, and G acts on P, there is a natural action of G on N. However, in general this action is *not* by automorphisms of $\mathbb{C}^m$. That is, N is the image of $\Phi : P \times (-\epsilon, \epsilon) \to \mathbb{C}^m$ and in general there is no G-action on $\mathbb{C}^m$ such that Φ is G-equivariant.

Nor does G act nontrivially on the set of SL m-folds N in $\mathbb{C}^m$ constructed from (P, χ). Instead, we should regard G as acting on the set of *parametrizations* Φ of N constructed in §3, so that one SL m-fold N will arise from the construction with many different parametrizations Φ, related by G.

Here is another way to say this. The maps Φ were constructed as solutions of an o.d.e. (2), with initial data ϕ_0 in a set $\mathcal{C}_P$ given in Definitions 3.2 and 3.4. It turns out that G acts naturally on $\mathcal{C}_P$, and two sets of initial data in the same G-orbit in $\mathcal{C}_P$ yield the same SL m-fold N in $\mathbb{C}^m$.

We shall use these ideas to predict the dimension of the family $\mathcal{M}_{(P,\chi)}$ of distinct SL m-folds N in $\mathbb{C}^m$ constructed from (P, χ) in §3. Suppose as above that P is a G-orbit, and define G' to be the Lie group of linear (or affine) automorphisms of $\mathbb{R}^n$ preserving P, and preserving χ up to scale. Then G is a subgroup of G', but may not be the whole thing.

For instance, in Example 4.3 P is invariant under *dilations* $\mathbf{x} \mapsto t\mathbf{x}$ in $\mathbb{R}^m$ for $t > 0$, which do not lie in G for $t \neq 1$, and multiply χ by t^{m-2}. In this case G is the identity component of $\mathrm{SO}(a, m-a)$, and the identity component of G' is $G \times \mathbb{R}_+$, that is, G together with the dilations. In [8] we will give other examples where G needs to be augmented by a 'dilation' group, which acts in a more complex way on $\mathbb{R}^n$.

We construct N from the integral curve of an o.d.e. in $\mathcal{C}_P$. The set of such curves has dimension $\dim \mathcal{C}_P - 1$. Two curves give the same SL m-fold N if they are equivalent under the action of G' on $\mathcal{C}_P$. Supposing that G' acts locally freely on $\mathcal{C}_P$, we guess that $\dim \mathcal{M}_{(P,\chi)} = \dim \mathcal{C}_P - 1 - \dim G'$.

In doing this calculation we have factored out the 'internal' symmetry group G' of the construction, which acts on the data used in the construction, but not on the set of SL m-folds we construct. However, there still remains the 'external' symmetry group of automorphisms of $\mathbb{C}^m$, which is $\mathrm{SU}(m)$ in the linear case (where the origin is a privileged point) and $\mathrm{SU}(m) \ltimes \mathbb{C}^m$ in the affine case.

Thus, if generic m-folds in $\mathcal{M}_{(P,\chi)}$ have no continuous symmetries, then the moduli space of SL m-folds up to automorphisms of $\mathbb{C}^m$ has dimension $\dim \mathcal{M}_{(P,\chi)} - m^2 + 1$ in the linear case, and $\dim \mathcal{M}_{(P,\chi)} - m^2 - 2m + 1$ in the affine case. This is probably the best measure of the number of 'interesting parameters' in the construction, once all symmetries are taken into account.

5 Examples from evolving centred quadrics

We will now apply the construction of §3 to the family of sets of linear evolution data (P,χ) defined using centred quadrics in $\mathbb{R}^m$ in Examples 4.2 and 4.3. In §5.1 we reduce the problem to an o.d.e. in complex functions $w_1,\ldots,w_m$ of a real variable t, and in §5.2 we rewrite the o.d.e. in terms of functions u,θ and $\theta_1,\ldots,\theta_m$ of t. Then in §5.3 we solve the equations explicitly, as far as we can; the solutions are written in terms of *elliptic integrals*.

Section 5.4 considers global properties of the solutions, and describes the resulting SL m-folds in four different cases. Finally, §5.5 considers one particularly interesting case in which the time evolution may be *periodic* in t, and investigates the conditions for periodicity.

It turns out that in the case that P is a sphere $\mathcal{S}^{m-1}$ in $\mathbb{R}^m$, the SL m-folds we construct have already been found using a different method by Lawlor [12], and completed by Harvey [4, p. 139–143]. Lawlor used his examples to prove the *angle conjecture*, a result on when the union of two m-planes in $\mathbb{R}^n$ is area-minimizing. The other cases of this section can also be studied using Lawlor and Harvey's method, and may well be known to them, but the author has not found the other cases published anywhere.

Much of this section runs parallel to the construction of $\mathrm{U}(1)^{m-2}$-invariant special Lagrangian cones in $\mathbb{C}^m$ in [7, §7] and uses the same ideas, because the o.d.e.s involved are very similar. However, the geometric interpretations are significantly different.

5.1 Reduction of the problem to an o.d.e.

Let $1\leqslant a\leqslant m$ and $c\in\mathbb{R}$, with $c>0$ if $a=m$, and define P and χ by

$$P=\bigl\{(x_1,\ldots,x_m)\in\mathbb{R}^m: x_1^2+\cdots+x_a^2-x_{a+1}^2-\cdots-x_m^2=c\bigr\}, \tag{5}$$

$$\begin{aligned}\chi=2\sum_{j=1}^{a}&(-1)^{j-1}x_j\,e_1\wedge\cdots\wedge e_{j-1}\wedge e_{j+1}\wedge\cdots\wedge e_m\\ -2\sum_{j=a+1}^{m}&(-1)^{j-1}x_j\,e_1\wedge\cdots\wedge e_{j-1}\wedge e_{j+1}\wedge\cdots\wedge e_m,\end{aligned} \tag{6}$$

where $e_j=\frac{\partial}{\partial x_j}$. Then (P,χ) is a set of linear evolution data. Consider linear maps $\phi:\mathbb{R}^m\rightarrow\mathbb{C}^m$ of the form

$$\phi:(x_1,\ldots,x_m)\mapsto(w_1x_1,\ldots,w_mx_m)\quad\text{for } w_1,\ldots,w_m \text{ in } \mathbb{C}\setminus\{0\}. \tag{7}$$

Then ϕ is injective and $\operatorname{Im}\phi$ is a Lagrangian m-plane in $\mathbb{C}^m$, so that ϕ lies in the subset $\mathcal{C}_P$ of $\operatorname{Hom}(\mathbb{R}^m,\mathbb{C}^m)$ given in Definition 3.2.

We will see that the evolution equation (2) for ϕ in $\mathcal{C}_P$ preserves ϕ of the form (7). So, consider a 1-parameter family $\bigl\{\phi_t:t\in(-\epsilon,\epsilon)\bigr\}$ given by

$$\phi_t:(x_1,\ldots,x_m)\mapsto\bigl(w_1(t)x_1,\ldots,w_m(t)x_m\bigr), \tag{8}$$

where $w_1, \dots, w_m$ are differentiable functions from $(-\epsilon, \epsilon)$ to $\mathbb{C} \setminus \{0\}$. We shall rewrite (2) as a first-order o.d.e. upon $w_1, \dots, w_m$.

Now $(\phi_t)_*(e_j) = w_j \frac{\partial}{\partial z_j} + \bar{w}_j \frac{\partial}{\partial \bar{z}_j}$. It is convenient to get rid of the $\bar{z}_j$ term by taking the (1,0)-component, giving $(\phi_t)_*(e_j)^{(1,0)} = w_j \frac{\partial}{\partial z_j}$. In the same way, from (6) the $(m-1,0)$ component $(\phi_t)_*(\chi)^{(m-1,0)}$ of $(\phi_t)_*(\chi)$ is

$$2\sum_{j=1}^{a}(-1)^{j-1}x_j w_1 \cdots w_{j-1}w_{j+1}\cdots w_m \frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_{j-1}}\wedge\frac{\partial}{\partial z_{j+1}}\wedge\cdots\wedge\frac{\partial}{\partial z_m}$$
$$-2\sum_{j=a+1}^{m}(-1)^{j-1}x_j w_1 \cdots w_{j-1}w_{j+1}\cdots w_m \frac{\partial}{\partial z_1}\wedge\cdots\wedge\frac{\partial}{\partial z_{j-1}}\wedge\frac{\partial}{\partial z_{j+1}}\wedge\cdots\wedge\frac{\partial}{\partial z_m}.$$

As Ω is an $(m,0)$-tensor, we see that the contraction of $(\phi_t)_*(\chi)$ with Ω is the same as that of $(\phi_t)_*(\chi)^{(m-1,0)}$ with Ω. Hence, using the index notation for tensors on $\mathbb{C}^m$, we get

$$(\phi_t)_*(\chi(x))^{a_1\dots a_{m-1}}\Omega_{a_1\dots a_{m-1}a_m} =$$
$$2\sum_{j=1}^{a} x_j w_1\cdots w_{j-1}w_{j+1}\cdots w_m(\mathrm{d}z_j)_{a_m} - 2\sum_{j=a+1}^{m} x_j w_1\cdots w_{j-1}w_{j+1}\cdots w_m(\mathrm{d}z_j)_{a_m}.$$

Hence

$$(\phi_t)_*(\chi(x))^{a_1\dots a_{m-1}}\Omega_{a_1\dots a_{m-1}a_m}g^{a_m b} =$$
$$2\sum_{j=1}^{a} x_j w_1\cdots w_{j-1}w_{j+1}\cdots w_m\Bigl(\frac{\partial}{\partial \bar{z}_j}\Bigr)^b - 2\sum_{j=a+1}^{m} x_j w_1\cdots w_{j-1}w_{j+1}\cdots w_m\Bigl(\frac{\partial}{\partial \bar{z}_j}\Bigr)^b.$$

Since $(\phi_t)_*(\chi(x))$ and g are real tensors, taking real parts gives

$$(\phi_t)_*(\chi(x))^{a_1\dots a_{m-1}}(\mathrm{Re}\,\Omega)_{a_1\dots a_{m-1}a_m}g^{a_m b} =$$
$$\sum_{j=1}^{a} x_j \bar{w}_1\cdots \bar{w}_{j-1}\bar{w}_{j+1}\cdots \bar{w}_m\Bigl(\frac{\partial}{\partial z_j}\Bigr)^b - \sum_{j=a+1}^{m} x_j \bar{w}_1\cdots \bar{w}_{j-1}\bar{w}_{j+1}\cdots \bar{w}_m\Bigl(\frac{\partial}{\partial z_j}\Bigr)^b$$
$$+\sum_{j=1}^{a} x_j w_1\cdots w_{j-1}w_{j+1}\cdots w_m\Bigl(\frac{\partial}{\partial \bar{z}_j}\Bigr)^b - \sum_{j=a+1}^{m} x_j w_1\cdots w_{j-1}w_{j+1}\cdots w_m\Bigl(\frac{\partial}{\partial \bar{z}_j}\Bigr)^b.$$

Now from (2) each side of this equation is $\bigl(\frac{\mathrm{d}\phi_t(x)}{\mathrm{d}t}\bigr)^b$, which satisfies

$$\Bigl(\frac{\mathrm{d}\phi_t(x)}{\mathrm{d}t}\Bigr)^b = \sum_{j=1}^{m} x_j \frac{\mathrm{d}w_j}{\mathrm{d}t}\Bigl(\frac{\partial}{\partial z_j}\Bigr)^b + \sum_{j=1}^{m} x_j \frac{\mathrm{d}\bar{w}_j}{\mathrm{d}t}\Bigl(\frac{\partial}{\partial \bar{z}_j}\Bigr)^b$$

by (8). Equating coefficients in the last two equations gives

$$\frac{\mathrm{d}w_j}{\mathrm{d}t} = \begin{cases} \overline{w_1\cdots w_{j-1}w_{j+1}\cdots w_m}, & j = 1,\dots,a, \\ -\overline{w_1\cdots w_{j-1}w_{j+1}\cdots w_m}, & j = a+1,\dots,m. \end{cases}$$

This is the first-order o.d.e. upon $w_1, \dots, w_m$ that we seek. Applying Theorem 3.3, we have proved:

Theorem 5.1 *Let $1 \leqslant a \leqslant m$ and $c \in \mathbb{R}$, with $c > 0$ if $a = m$. Suppose $w_1, \ldots, w_m$ are differentiable functions $w_j : (-\epsilon, \epsilon) \to \mathbb{C} \setminus \{0\}$ satisfying*

$$\frac{\mathrm{d}w_j}{\mathrm{d}t} = \begin{cases} \overline{w_1 \cdots w_{j-1} w_{j+1} \cdots w_m}, & j = 1, \ldots, a, \\ -\overline{w_1 \cdots w_{j-1} w_{j+1} \cdots w_m}, & j = a+1, \ldots, m. \end{cases} \tag{9}$$

Define a subset N of $\mathbb{C}^m$ by

$$N = \Big\{ \big(w_1(t)x_1, \ldots, w_m(t)x_m\big) : t \in (-\epsilon, \epsilon), \quad x_j \in \mathbb{R}, \\ x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - \cdots - x_m^2 = c \Big\}. \tag{10}$$

Then N is a special Lagrangian submanifold in $\mathbb{C}^m$.

Observe that (9) agrees with [7, eq. (8)], with $a_j = 1$ for $j \leqslant a$ and $a_j = -1$ for $j > a$. Thus, we can follow the analysis of [7, §7] to understand the solutions of (9). Furthermore, we showed in [7, §7.6] that [7, eq. (8)] is a *completely integrable Hamiltonian system*, and the proof also applies to (9).

5.2 Rewriting these equations

We now rewrite Theorem 5.1 using different variables. If $j \leqslant a$ then (9) gives

$$\frac{\mathrm{d}|w_j|^2}{\mathrm{d}t} = w_j \frac{\mathrm{d}\bar{w}_j}{\mathrm{d}t} + \bar{w}_j \frac{\mathrm{d}w_j}{\mathrm{d}t} = w_1 \cdots w_m + \overline{w_1 \cdots w_m} = 2\,\mathrm{Re}(w_1 \cdots w_m),$$

and in the same way we get

$$\frac{\mathrm{d}|w_j|^2}{\mathrm{d}t} = \begin{cases} 2\,\mathrm{Re}(w_1 \cdots w_m), & j = 1, \ldots, a, \\ -2\,\mathrm{Re}(w_1 \cdots w_m), & j = a+1, \ldots, m. \end{cases} \tag{11}$$

Let $\lambda \in \mathbb{R}$ be a constant, to be chosen later. Define $\alpha_1, \ldots, \alpha_m$ by

$$\alpha_j = \begin{cases} |w_j(0)|^2 - \lambda, & j = 1, \ldots, a, \\ |w_j(0)|^2 + \lambda, & j = a+1, \ldots, m, \end{cases} \tag{12}$$

and a function $u : (-\epsilon, \epsilon) \to \mathbb{R}$ by

$$u(t) = \lambda + 2 \int_0^t \mathrm{Re}\big(w_1(s) \cdots w_m(s)\big) \mathrm{d}s,$$

so that $u(0) = \lambda$. Then (11) gives

$$|w_j|^2 = \begin{cases} \alpha_j + u, & j = 1, \ldots, a, \\ \alpha_j - u, & j = a+1, \ldots, m. \end{cases} \tag{13}$$

Thus we may write

$$w_j(t) = \begin{cases} \mathrm{e}^{i\theta_j(t)} \sqrt{\alpha_j + u(t)}, & j = 1, \ldots, a, \\ \mathrm{e}^{i\theta_j(t)} \sqrt{\alpha_j - u(t)}, & j = a+1, \ldots, m, \end{cases}$$

for differentiable functions $\theta_1, \ldots, \theta_m : (-\epsilon, \epsilon) \to \mathbb{R}$.

Define

$$\theta = \theta_1 + \cdots + \theta_m \quad \text{and} \quad Q(u) = \prod_{j=1}^{a} (\alpha_j + u) \prod_{j=a+1}^{m} (\alpha_j - u).$$

Then we see that

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 2\,\mathrm{Re}(w_1 \cdots w_m) = 2Q(u)^{1/2} \cos\theta.$$

Furthermore, expanding out (9) shows that

$$\frac{\mathrm{d}\theta_j}{\mathrm{d}t} = \begin{cases} -\dfrac{Q(u)^{1/2}\sin\theta}{\alpha_j + u}, & j = 1, \ldots, a, \\ \dfrac{Q(u)^{1/2}\sin\theta}{\alpha_j - u}, & j = a+1, \ldots, m. \end{cases}$$

Summing this equation from $j = 1$ to m gives

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\,Q(u)^{1/2}\sin\theta \left(\sum_{j=1}^{a} \frac{1}{\alpha_j + u} - \sum_{j=a+1}^{m} \frac{1}{\alpha_j - u} \right).$$

Thus, we may rewrite Theorem 5.1 in the following way.

Theorem 5.2 *Let u and $\theta_1, \ldots, \theta_m$ be differentiable functions $(-\epsilon, \epsilon) \to \mathbb{R}$ satisfying*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 2Q(u)^{1/2} \cos\theta \tag{14}$$

$$\text{and} \quad \frac{\mathrm{d}\theta_j}{\mathrm{d}t} = \begin{cases} -\dfrac{Q(u)^{1/2}\sin\theta}{\alpha_j + u}, & j = 1, \ldots, a, \\ \dfrac{Q(u)^{1/2}\sin\theta}{\alpha_j - u}, & j = a+1, \ldots, m, \end{cases} \tag{15}$$

where $\theta = \theta_1 + \cdots + \theta_m$, so that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\,Q(u)^{1/2}\sin\theta \left(\sum_{j=1}^{a} \frac{1}{\alpha_j + u} - \sum_{j=a+1}^{m} \frac{1}{\alpha_j - u} \right). \tag{16}$$

Suppose that $\alpha_j + u > 0$ for $j = 1, \ldots, a$ and $\alpha_j - u > 0$ for $j = a+1, \ldots, m$ and $t \in (-\epsilon, \epsilon)$. Define a subset N of $\mathbb{C}^m$ to be

$$\begin{aligned} N = \Big\{ \big(& x_1 \mathrm{e}^{i\theta_1(t)} \sqrt{\alpha_1 + u(t)}, \ldots, x_a \mathrm{e}^{i\theta_a(t)} \sqrt{\alpha_a + u(t)}, \\ & x_{a+1} \mathrm{e}^{i\theta_{a+1}(t)} \sqrt{\alpha_{a+1} - u(t)}, \ldots, x_m \mathrm{e}^{i\theta_m(t)} \sqrt{\alpha_m - u(t)} \ \big) : \\ & t \in (-\epsilon, \epsilon), \ \ x_j \in \mathbb{R}, \ \ x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - \cdots - x_m^2 = c \Big\}. \end{aligned} \tag{17}$$

Then N is a special Lagrangian submanifold in $\mathbb{C}^m$.

Now (14) and (16) give $\frac{\mathrm{d}u}{\mathrm{d}t}$ and $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ as functions of u and θ. Dividing one by the other gives an expression for $\frac{\mathrm{d}u}{\mathrm{d}\theta}$, eliminating t. Suppose for the moment that $\sin(\theta(0)) \neq 0$. Then separating variables gives

$$\int_{u(0)}^{u(t)} \left(\sum_{j=1}^{a} \frac{1}{\alpha_j + u} - \sum_{j=a+1}^{m} \frac{1}{\alpha_j - u} \right) \mathrm{d}u = -2 \int_{\theta(0)}^{\theta(t)} \cot\theta \, \mathrm{d}\theta,$$

which integrates explicitly to

$$\log Q(u) = -2 \log \sin\theta + C$$

for all $t \in (-\epsilon, \epsilon)$, for some $C \in \mathbb{R}$. Exponentiating gives $Q(u)\sin^2\theta \equiv \mathrm{e}^C > 0$.

If on the other hand $\sin\theta(0) = 0$ then (16) shows that θ is constant in $(-\epsilon, \epsilon)$, so $Q(u)\sin^2\theta \equiv 0$. In both cases $Q(u)\sin^2\theta$ is constant, so its square root $Q(u)^{1/2}\sin\theta$ is also constant, as it is continuous. Thus we have $Q(u)^{1/2}\sin\theta \equiv A$ for some $A \in \mathbb{R}$.

This simplifies (15) and (16), as we can replace the factor $Q(u)^{1/2}\sin\theta$ by A. Also, from (14) we find that

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 = 4Q(u)\cos^2\theta = 4\bigl(Q(u) - Q(u)\sin^2\theta\bigr) = 4\bigl(Q(u) - A^2\bigr).$$

Thus we have proved the following analogue of [7, Prop. 7.3]:

Proposition 5.3 *In the situation of Theorem 5.2 we have*

$$Q(u)^{1/2}\sin\theta \equiv A \tag{18}$$

for some $A \in \mathbb{R}$ *and all* $t \in (-\epsilon, \epsilon)$, *and* (14)–(16) *are equivalent to*

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 = 4\bigl(Q(u) - A^2\bigr), \tag{19}$$

$$\frac{\mathrm{d}\theta_j}{\mathrm{d}t} = \begin{cases} -\dfrac{A}{\alpha_j + u}, & j = 1, \ldots, a, \\ \dfrac{A}{\alpha_j - u}, & j = a+1, \ldots, m, \end{cases} \tag{20}$$

and $$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -A \left(\sum_{j=1}^{a} \frac{1}{\alpha_j + u} - \sum_{j=a+1}^{m} \frac{1}{\alpha_j - u} \right). \tag{21}$$

5.3 Explicit solution using elliptic integrals

Next we will write down the SL m-fold N of Theorem 5.2 in a more simple and explicit way. A nice way of doing this is to eliminate t, and write everything instead as a function of u. Now $\frac{\mathrm{d}u}{\mathrm{d}t}$ has the same sign as $\cos\theta$ by (14). Thus,

if $\cos\theta$ changes sign in $(-\epsilon,\epsilon)$ then we cannot write t as a function of u, but if $\cos\theta$ has constant sign then we can.

Let us assume that $\theta(t)\in(-\pi/2,\pi/2)$ for all $t\in(-\epsilon,\epsilon)$, so that $\cos\theta$ is positive. Then (19) gives $\frac{\mathrm{d}u}{\mathrm{d}t}=2\sqrt{Q(u)-A^2}$, and integrating gives

$$\int_{u(0)}^{u(t)}\frac{\mathrm{d}u}{2\sqrt{Q(u)-A^2}}=\int_0^t\mathrm{d}t=t.$$

This defines u implicitly as a function of t. From (19) and (20) we get

$$\frac{\mathrm{d}\theta_j}{\mathrm{d}u}=\begin{cases}-\dfrac{A}{2(\alpha_j+u)\sqrt{Q(u)-A^2}}, & j=1,\ldots,a,\\ \dfrac{A}{2(\alpha_j-u)\sqrt{Q(u)-A^2}}, & j=a{+}1,\ldots,m.\end{cases}$$

Integrating these gives expressions for θ_j in terms of u, and we have proved:

Theorem 5.4 *Suppose $\theta(t)\in(-\pi/2,\pi/2)$ for all $t\in(-\epsilon,\epsilon)$. Then the special Lagrangian m-fold N of Theorem 5.2 is given explicitly by*

$$\begin{aligned}N=\Bigl\{&\bigl(x_1\mathrm{e}^{i\theta_1(u)}\sqrt{\alpha_1+u},\ldots,x_a\mathrm{e}^{i\theta_a(u)}\sqrt{\alpha_a+u},\\ &x_{a+1}\mathrm{e}^{i\theta_{a+1}(u)}\sqrt{\alpha_{a+1}-u},\ldots,x_m\mathrm{e}^{i\theta_m(u)}\sqrt{\alpha_m-u}\,\bigr):\\ &u\in\bigl(u(-\epsilon),u(\epsilon)\bigr),\;\; x_j\in\mathbb{R},\;\; x_1^2+\cdots+x_a^2-x_{a+1}^2-\cdots-x_m^2=c\Bigr\},\end{aligned}$$

where the functions $\theta_j(u)$ are given by

$$\theta_j(u)=\begin{cases}\theta_j\bigl(u(0)\bigr)-\dfrac{A}{2}\displaystyle\int_{u(0)}^{u}\frac{\mathrm{d}v}{(\alpha_j+v)\sqrt{Q(v)-A^2}}, & j=1,\ldots,a,\\ \theta_j\bigl(u(0)\bigr)+\dfrac{A}{2}\displaystyle\int_{u(0)}^{u}\frac{\mathrm{d}v}{(\alpha_j-v)\sqrt{Q(v)-A^2}}, & j=a{+}1,\ldots,m.\end{cases}$$

5.4 A qualitative description of the solutions

We now describe the SL m-folds N in $\mathbb{C}^m$ emerging from the construction of Theorem 5.2, dividing into four cases, depending on the values of A and a.

Case (a): $A=0$.

When $A=0$, we see from (20) that $\theta_1,\ldots,\theta_m$ are constant with $\theta_1+\cdots+\theta_m=n\pi$ for some $n\in\mathbb{Z}$, and the SL m-fold N of (17) is a subset of the special Lagrangian m-plane

$$\bigl\{(x_1\mathrm{e}^{i\theta_1},\ldots,x_m\mathrm{e}^{i\theta_m}):x_1,\ldots,x_m\in\mathbb{R}\bigr\}.$$

Thus the case $A = 0$ is not very interesting. If we replace θ_1 by $\theta_1 + \pi$ and t by $-t$ then A changes sign, but the manifold N of (17) is unchanged. So we may assume in the remaining cases that $A > 0$. Then it turns out that the equations behave very differently depending on whether $a = m$ or $a < m$. We consider the $a = m$ case first.

Case (b): $a = m$**,** $c > 0$ **and** $A > 0$**.**

This case has already been studied by Lawlor [12] and Harvey [4, p. 139–143], using somewhat different methods. After some changes of notation, one can show that Harvey [4, Th. 7.78, p. 140] is equivalent to the case $a = m$, $c > 0$ of Theorem 5.4. Lawlor used his examples to prove the *angle conjecture*, a result on when the union of two m-planes in $\mathbb{R}^n$ is area-minimizing. When $\alpha_1 = \cdots = \alpha_m$, the manifolds are $\mathrm{SO}(m)$-invariant, and are given in [5, §III.3.B].

When $m \geqslant 3$, it can be shown that equation (9) admits solutions on a bounded open interval (γ, δ) with $\gamma < 0 < \delta$, such that $u(t) \to \infty$ as $t \to \gamma_+$ and $t \to \delta_-$, so that the solutions cannot be extended continuously outside (γ, δ). When $m = 2$, solutions exist on $\mathbb{R}$, with $u(t) \to \infty$ as $t \to \pm\infty$, so we can put '$\gamma = -\infty$' and '$\delta = \infty$' in this case.

The SL m-fold N defined using the full solution interval (γ, δ) is a closed, embedded special Lagrangian m-fold diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}$. It is the total space of a family of ellipsoids P_t in $\mathbb{C}^m$, parametrized by t. As t approaches γ or δ these ellipsoids go to infinity in $\mathbb{C}^m$, and also become more and more spherical.

At infinity, N is asymptotic to order r^{1-m} to the union of two special Lagrangian m-planes $\mathbb{R}^m$ in $\mathbb{C}^m$ meeting at 0, and we can think of N as a *connected sum* of two copies of $\mathbb{R}^m$. These examples are interesting because they provide local models for the creation of new SL m-folds in Calabi–Yau m-folds as connected sums of other SL m-folds, as in [6, §6–§7] when $m = 3$.

It remains to consider the cases in which $A > 0$ and $1 \leqslant a \leqslant m - 1$. Recall that the definition (12) of $\alpha_1, \ldots, \alpha_m$ depended on an arbitrary constant $\lambda \in \mathbb{R}$. It is easy to show that there exists a unique $\lambda \in \mathbb{R}$ such that $\alpha_j > 0$ for all j, and

$$\sum_{j=1}^{a} \frac{1}{\alpha_j} = \sum_{j=a+1}^{m} \frac{1}{\alpha_j}. \tag{22}$$

Let us choose this value of λ.

Since $Q(u) = |w_1|^2 \cdots |w_m|^2$, and $Q(u) \geqslant A^2 > 0$ as $A > 0$, we have $|w_j|^2 > 0$ for all j. Thus, from (13) we see that $u(t)$ is confined to the open interval

$$\big(-\min_{1 \leqslant j \leqslant a} \alpha_j, \min_{a+1 \leqslant j \leqslant m} \alpha_j\big) \tag{23}$$

for all t for which the solution exists. It follows from (22) that $Q'(0) = 0$. As the roots $-\alpha_1, \ldots, -\alpha_a, \alpha_{a+1}, \ldots, \alpha_m$ of $Q(u)$ are all real and none lie in (23), zero is the only turning point of Q in the interval (23). Thus, Q achieves its maximum in (23) at 0, and $Q(0) = \alpha_1 \cdots \alpha_m$.

But $Q(u) \geqslant A^2$ by (18). Hence, for all t we have

$$0 < A^2 \leqslant Q(u) \leqslant \alpha_1 \cdots \alpha_m. \tag{24}$$

In particular, this shows that $A \leqslant (\alpha_1 \cdots \alpha_m)^{1/2}$. We shall divide into two more cases, depending on whether $A = (\alpha_1 \cdots \alpha_m)^{1/2}$ or $A < (\alpha_1 \cdots \alpha_m)^{1/2}$.

Case (c): $1 \leqslant a \leqslant m-1$ **and** $A = (\alpha_1 \cdots \alpha_m)^{1/2}$**.**

In this case, (24) gives $\alpha_1 \cdots \alpha_m \leqslant Q(u) \leqslant \alpha_1 \cdots \alpha_m$, so $Q(u) \equiv \alpha_1 \cdots \alpha_m$. It easily follows that $u \equiv 0$, $\cos\theta \equiv 0$ and $\sin\theta \equiv 1$, so that $\theta \equiv (2n + \frac{1}{2})\pi$ for some $n \in \mathbb{Z}$. Equation (20) then gives

$$\theta_j(t) = \begin{cases} \theta_j(0) - At/\alpha_j & j = 1, \ldots, a, \\ \theta_j(0) + At/\alpha_j & j = a+1, \ldots, m. \end{cases}$$

Thus solutions exist for all $t \in \mathbb{R}$. Define

$$a_j = \begin{cases} -A/\alpha_j & j = 1, \ldots, a, \\ \ \ A/\alpha_j & j = a+1, \ldots, m, \end{cases} \quad \text{and} \quad y_j = \alpha_j^{1/2} x_j, \quad j = 1, \ldots, m.$$

Then we find that $a_1 + \cdots + a_m = 0$, and N is given by

$$\Big\{ \big(\mathrm{e}^{i(\theta_1(0) + a_1 t)} y_1, \ldots, \mathrm{e}^{i(\theta_m(0) + a_m t)} y_m \big) : t \in \mathbb{R}, \quad y_j \in \mathbb{R}, \\ a_1 y_1^2 + \cdots + a_m y_m^2 = -Ac \Big\}.$$

Now apart from the constant phase factors $\mathrm{e}^{i\theta_j(0)}$, this is one of the SL m-folds constructed in [7, Prop. 9.3] using the 'perpendicular symmetry' idea of [7, §9], with $n = m$ and $G = \mathrm{U}(1)$ or $\mathbb{R}$. When $a_1, \ldots, a_m$ are integers, this example is discussed in [7, Ex. 9.4].

Case (d): $1 \leqslant a \leqslant m-1$ **and** $0 < A < (\alpha_1 \cdots \alpha_m)^{1/2}$**.**

This is very similar to case (c) of [7, §7], and following the proof of [7, Prop. 7.11] we can show:

Proposition 5.5 *Suppose* $1 \leqslant a \leqslant m-1$*, and* $\alpha_1, \ldots, \alpha_m$ *satisfy* $\alpha_j > 0$ *and* (22)*. Let* $u(0)$ *and* $\theta_1(0), \ldots, \theta_m(0)$ *be given, such that* $\alpha_j + u(0) > 0$ *for* $j = 1, \ldots, a$ *and* $\alpha_j - u(0) > 0$ *for* $j = a+1, \ldots, m$*, and*

$$0 < A = Q(u(0))^{1/2} \sin\theta(0) < (\alpha_1 \cdots \alpha_m)^{1/2},$$

where $\theta(0) = \theta_1(0) + \cdots + \theta_m(0)$*. Then there exist unique solutions* $u(t)$*,* $\theta_j(t)$ *and* $\theta(t)$ *to equations* (14)–(16) *of Theorem 5.2 for all* $t \in \mathbb{R}$*, with these values at* $t = 0$*. Furthermore* u *and* θ *are nonconstant and periodic with period* $T > 0$*, and there exist* $\beta_1, \cdots \beta_m \in \mathbb{R}$ *with* $\beta_j < 0$ *when* $j = 1, \ldots, a$ *and* $\beta_j > 0$ *when* $j = a+1, \ldots, m$ *and* $\beta_1 + \cdots + \beta_m = 0$*, such that* $\theta_j(t+T) = \theta_j(t) + \beta_j$ *for* $j = 1, \ldots, m$ *and all* $t \in \mathbb{R}$*.*

Here solutions u, θ to equations (14) and (16) are periodic with period T, just as in [7, §7.5]. Therefore $\frac{\mathrm{d}\theta_j}{\mathrm{d}t}$ is periodic with period T by (15), which implies that $\theta_j(t+T) = \theta_j(t) + \beta_j$ for some $\beta_j \in \mathbb{R}$. But $\frac{\mathrm{d}\theta_j}{\mathrm{d}t} < 0$ for $j \leqslant a$ and $\frac{\mathrm{d}\theta_j}{\mathrm{d}t} > 0$ for $j > a$ by (20), so that $\beta_j < 0$ when $j \leqslant a$ and $\beta_j > 0$ when $j > a$. Also $\theta = \theta_1 + \cdots + \theta_m$, so that $\theta(t+T) = \theta(t) + \beta_1 + \cdots + \beta_m$. As θ is periodic with period T, we see that $\beta_1 + \cdots + \beta_m = 0$.

What this means is that when t goes through one cycle of length T, the complex coordinates $z_1, \ldots, z_m$ don't return to their starting points, but instead are taken to $\mathrm{e}^{i\beta_1} z_1, \ldots, \mathrm{e}^{i\beta_m} z_m$.

5.5 Periodic solutions in case (d)

We have seen that in case (d) above, u and θ are periodic functions with period T, but $\theta_1, \ldots, \theta_m$ are not periodic, and satisfy $\theta_j(t+T) = \theta_j(t) + \beta_j$ for $\beta_1, \ldots, \beta_m$ real numbers with $\beta_j < 0$ if $j \leqslant a$ and $\beta_j > 0$ if $j > a$. But N in (17) depends only on $\mathrm{e}^{i\theta_j}$ rather than on θ_j, so that $\theta_1, \ldots, \theta_m$ matter only up to multiples of 2π.

Thus, if $\beta_1, \ldots, \beta_m$ are integer multiples of 2π, then the evolution defining N repeats after time T. Actually it's enough for β_j to be multiples of π, as we can change the sign of x_j in (17). More generally, if $\beta_1, \ldots, \beta_m$ are rational multiples of π, then the evolution repeats after time nT, where $n > 0$ is the lowest common multiple of the denominators of the rational factors.

For our later applications, these periodic solutions are more interesting than the non-periodic ones, because they give rise to closed special Lagrangian m-folds in $\mathbb{C}^m$ that can be local models for singularities of special Lagrangian m-folds in Calabi–Yau manifolds. But the non-periodic solutions are not closed in $\mathbb{C}^m$, and are not suitable as local models in the same way.

Therefore we will study the dependence of the β_j upon the initial data. It is easy to see that $\beta_1, \ldots, \beta_m$ depend only on $a, m, \alpha_1, \ldots, \alpha_m$ and A, and not on $u(0)$ or $\theta_1(0), \ldots, \theta_m(0)$. Also, from above the α_j and A satisfy

$$\alpha_j > 0, \quad \sum_{j=1}^{a} \frac{1}{\alpha_j} = \sum_{j=a+1}^{m} \frac{1}{\alpha_j} \quad \text{and} \quad 0 < A < (\alpha_1 \cdots \alpha_m)^{1/2}. \tag{25}$$

Given any $\alpha_1, \ldots, \alpha_m$ and A satisfying these conditions, there exists a set of initial data $u(0), \theta_1(0), \ldots, \theta_m(0)$ with these values. For instance, we can fix $u(0) = 0$, and then take any $\theta_1(0), \ldots, \theta_m(0)$ such that $\sin\theta(0) = A(\alpha_1 \cdots \alpha_m)^{-1/2}$.

Consider what happens to the data when we rescale N by a constant factor $\kappa > 0$. Calculation shows that we should replace u, θ_j, α_j and A by u', θ'_j, α'_j and A', where

$$u'(t) = \kappa^2 u\bigl(\kappa^{m-2} t\bigr), \quad \theta'_j(t) = \theta_j\bigl(\kappa^{m-2} t\bigr), \quad \alpha'_j = \kappa^2 \alpha_j \quad \text{and} \quad A' = \kappa^m A.$$

These give new solutions to the equations with period $T' = \kappa^{2-m} T$, and unchanged values of $\beta_1, \ldots, \beta_m$. The corresponding SL m-fold N' is $\kappa N = \{\kappa \mathbf{z} : \mathbf{z} \in N\}$.

We can now do a parameter count. Since $\beta_1+\cdots+\beta_m=0$, there are only $m-1$ independent β_j. These depend on the $m+1$ variables $\alpha_1,\ldots,\alpha_m$ and A, which satisfy one equation $\sum_{j=1}^a \frac{1}{\alpha_j}=\sum_{j=a+1}^m \frac{1}{\alpha_j}$. Thus, the β_j can be regarded as $m-1$ functions of m variables. However, rescaling by κ leaves the β_j unchanged, but removes one degree of freedom from the α_j and A.

Thus, in the initial data $\alpha_1,\ldots,\alpha_m$ and A there are only $m-1$ interesting degrees of freedom, and there are $m-1$ independent β_j depending on them. The obvious conjecture is that these two sets of $m-1$ parameters correspond, and that the map from sets of α_j and A satisfying (25) and sets of β_j satisfying $\beta_1+\cdots+\beta_m=0$ is generically locally surjective, and locally injective modulo rescaling by $\kappa>0$ as above. In our next few results we shall show that this is true.

In the following proposition, modelled on [7, Prop. 7.13], we regard the α_j as fixed, and evaluate the limits of the β_j as $A\rightarrow 0$ and $A\rightarrow(\alpha_1\cdots\alpha_m)^{1/2}$. For simplicity we order the α_j so that $\alpha_1\leqslant\cdots\leqslant\alpha_a$ and $\alpha_{a+1}\leqslant\cdots\leqslant\alpha_m$.

Proposition 5.6 *Suppose* $1\leqslant a<m$ *and* $\alpha_1,\ldots,\alpha_m>0$ *satisfy*

$$\begin{gathered}\alpha_1=\cdots=\alpha_k<\alpha_{k+1}\leqslant\cdots\leqslant\alpha_a,\\ \alpha_{a+1}\leqslant\cdots\leqslant\alpha_{m-l}<\alpha_{m-l+1}=\cdots=\alpha_m \;\; \textit{and} \;\; \sum_{j=1}^a\frac{1}{\alpha_j}=\sum_{j=a+1}^m\frac{1}{\alpha_j}.\end{gathered} \tag{26}$$

Regarding $\alpha_1,\ldots,\alpha_m$ *as fixed and letting* A *vary in* $\bigl(0,(\alpha_1\cdots\alpha_m)^{1/2}\bigr)$, *we find that as* $A\rightarrow 0$, *we have*

$$\beta_j\rightarrow\begin{cases}-\dfrac{\pi}{k}, & 1\leqslant j\leqslant k,\\ 0, & k<j\leqslant m-l,\\ \dfrac{\pi}{l}, & m-l<j\leqslant m,\end{cases} \tag{27}$$

and as $A\rightarrow(\alpha_1\cdots\alpha_m)^{1/2}$, *we have*

$$\beta_j\rightarrow\begin{cases}-2\pi\alpha_j^{-1}\Bigl(2\sum_{i=1}^m\alpha_i^{-2}\Bigr)^{-1/2}, & 1\leqslant j\leqslant a,\\ 2\pi\alpha_j^{-1}\Bigl(2\sum_{i=1}^m\alpha_i^{-2}\Bigr)^{-1/2}, & a+1\leqslant j\leqslant m.\end{cases} \tag{28}$$

Proof. Let γ,δ be the minimum and maximum values of u. Then $-\alpha_1<\gamma<0<\delta<\alpha_m$ and $Q(\gamma)=Q(\delta)=A^2$, and using the ideas of §5.3 we find that

$$\beta_j=\begin{cases}-\displaystyle\int_\gamma^\delta\frac{\mathrm{d}v}{(\alpha_j+v)\sqrt{A^{-2}Q(v)-1}}, & j=1,\ldots,a,\\ \displaystyle\int_\gamma^\delta\frac{\mathrm{d}v}{(\alpha_j-v)\sqrt{A^{-2}Q(v)-1}}, & j=a+1,\ldots,m.\end{cases} \tag{29}$$

As $A \to 0$ we have $\gamma \to -\alpha_1$ and $\delta \to \alpha_m$. Also, the factors $(A^{-2}Q(v)-1)^{-1/2}$ in (29) tend to zero, except near γ and δ. Hence, as $A \to 0$, the integrands in (29) get large near $\gamma \approx -\alpha_1$ and $\delta \approx \alpha_m$, and very close to zero in between.

So to understand the β_j as $A \to 0$, it is enough to study the integrals (29) near γ and δ. We shall model them at γ. Then near $v = -\alpha_1$ we have

$$Q(v) \approx C(v+\alpha_1)^k, \qquad \text{where} \qquad C = \prod_{i=k+1}^{a} (\alpha_i - \alpha_1) \prod_{i=a+1}^{m} (\alpha_i + \alpha_1).$$

Since $A^2 = Q(\gamma)$ this gives $A^2 \approx C(\gamma+\alpha_1)^k$, so that $\gamma \approx A^{2/k}C^{-1/k} - \alpha_1$.

Therefore, when $v \approx \gamma$ we have $A^{-2}Q(v) - 1 \approx A^{-2}C(v+\alpha_1)^k - 1$, so when A is small and $j = 1, \ldots, k$ we have

$$\begin{aligned} \int_\gamma^0 \frac{\mathrm{d}v}{(\alpha_j+v)\sqrt{A^{-2}Q(v)-1}} &\approx \int_{A^{2/k}C^{-1/k}-\alpha_1}^0 \frac{\mathrm{d}v}{(\alpha_1+v)\sqrt{A^{-2}C(v+\alpha_1)^k-1}} \\ &\approx \int_0^\infty \frac{2\mathrm{d}w}{k(w^2+1)} = \frac{\pi}{k}, \end{aligned}$$

changing variables to $w = \sqrt{A^{-2}C(v+\alpha_1)^k - 1}$, where in the second line some surprising cancellations happen, and we have also approximated the upper limit $\sqrt{A^{-2}C\alpha_1^k - 1}$ by ∞.

When $k+1 \leqslant j \leqslant a$ and A is small we have

$$\int_\gamma^0 \frac{\mathrm{d}v}{(\alpha_j+v)\sqrt{A^{-2}Q(v)-1}} \approx \int_{A^{2/k}C^{-1/k}-\alpha_1}^0 \frac{\mathrm{d}v}{\alpha_j\sqrt{A^{-2}C(v+\alpha_1)^k-1}} \approx 0,$$

and similarly when $a+1 \leqslant j \leqslant m$ and A is small we have

$$\int_\gamma^0 \frac{\mathrm{d}v}{(\alpha_j-v)\sqrt{A^{-2}Q(v)-1}} \approx 0.$$

So on $[\gamma, 0]$ the integrals (29) are close to $-\pi/k$ for $1 \leqslant j \leqslant k$ and 0 for $j > k$ for small A. In the same way, on $[0, \delta]$ the integrals (29) are close to π/l for $m - l < j \leqslant m$ and 0 for $j \leqslant m - l$ for small A. This proves (27).

Next consider the behaviour of β_j as $A \to (\alpha_1 \cdots \alpha_m)^{1/2}$. When A is close to $(\alpha_1 \cdots \alpha_m)^{1/2}$, u is small and $\sin\theta$ close to 1, so θ remains close to $\pi/2$. Write $\theta = \frac{\pi}{2} + \phi$, for ϕ small. Then, setting $Q(u) \approx \alpha_1 \cdots \alpha_m$ and

$$\cos\theta \approx -\phi, \;\; \sin\theta \approx 1 \;\text{ and }\; \sum_{j=1}^{a} \frac{1}{\alpha_j+u} - \sum_{j=a+1}^{m} \frac{1}{\alpha_j-u} \approx -u \sum_{j=1}^{m} \alpha_j^{-2},$$

taking only the highest order terms, equations (14) and (16) become

$$\frac{\mathrm{d}u}{\mathrm{d}t} \approx -2(\alpha_1 \cdots \alpha_m)^{1/2}\phi \quad \text{and} \quad \frac{\mathrm{d}\phi}{\mathrm{d}t} \approx u(\alpha_1 \cdots \alpha_m)^{1/2} \sum_{j=1}^{m} \alpha_j^{-2},$$

so that u and θ undergo approximately simple harmonic oscillations with period $T=2\pi\bigl(2\alpha_1\cdots\alpha_m\sum_{j=1}^m\alpha_j^{-2}\bigr)^{-1/2}$. Then (15) shows that

$$\frac{\mathrm{d}\theta_j}{\mathrm{d}t}\approx\begin{cases}-\alpha_j^{-1}(\alpha_1\cdots\alpha_m)^{1/2}, & 1\leqslant j\leqslant a,\\ \alpha_j^{-1}(\alpha_1\cdots\alpha_m)^{1/2}, & a+1\leqslant j\leqslant m.\end{cases}$$

So $\beta_j\approx\frac{\mathrm{d}\theta_j}{\mathrm{d}t}T$, as $\frac{\mathrm{d}\theta_j}{\mathrm{d}t}$ is approximately constant. This proves (28). □

We can use these limits to show that the map $\boldsymbol\beta$ from $\alpha_1,\ldots,\alpha_m,A$ to $\beta_1,\ldots,\beta_m$ with $\beta_1+\cdots+\beta_m=0$ is generically locally surjective.

Proposition 5.7 *Regard $\boldsymbol\beta=(\beta_1,\ldots,\beta_m)$ as a function of $\bigl(\alpha_1,\ldots,\alpha_m,A\bigr)$. Then $\boldsymbol\beta$ is a real analytic map from U to V, where*

$$U=\Bigl\{(\alpha_1,\ldots,\alpha_m,A):\alpha_j>0,\ \sum_{j=1}^a\frac{1}{\alpha_j}=\sum_{j=a+1}^m\frac{1}{\alpha_j},\ 0<A<(\alpha_1\cdots\alpha_m)^{1/2}\Bigr\}$$
$$\textit{and}\quad V=\bigl\{(x_1,\ldots,x_m)\in(-\infty,0)^a\times(0,\infty)^{m-a}:x_1+\cdots+x_m=0\bigr\}.$$

When $m=2$ we have $\boldsymbol\beta(u)=(-\pi,\pi)$ for all $u\in U$. When $m\geqslant 3$, the image $\boldsymbol\beta(U)$ is $(m–1)$-dimensional, and for a dense open subset of $u\in U$ the derivative $\mathrm{d}\boldsymbol\beta|_u:\mathbb{R}^{m+1}\rightarrow\bigl\{(x_1,\ldots,x_m)\in\mathbb{R}^m:x_1+\cdots+x_m=0\bigr\}$ is surjective.

Proof. From §5.4 we know that $\beta_j<0$ when $j\leqslant a$ and $\beta_j>0$ when $j>a$, and $\beta_1+\cdots+\beta_m=0$, so that $\boldsymbol\beta$ does map U to V. As A,γ and δ are clearly real analytic functions of the α_j and A, we see from (29) that $\boldsymbol\beta$ is real analytic.

When $m=2$ we must have $a=1$, and going back to Theorem 5.1 we see that the equations on w_1,w_2 are $\frac{\mathrm{d}w_1}{\mathrm{d}t}=\bar w_2$ and $\frac{\mathrm{d}w_2}{\mathrm{d}t}=-\bar w_1$, which are *real linear*, and admit simple harmonic solutions with period 2π for any nonzero initial data. Translating this into the notation of Theorem 5.2, we find that u and θ have period π and $\beta_1=-\pi$, $\beta_2=\pi$ for any initial data in U.

Now the limits of $\beta_1,\ldots,\beta_m$ in (28) satisfy $\sum_{j=1}^m\beta_j^2=2\pi^2$. Thus, from (28) we see that the closure $\overline{\boldsymbol\beta(U)}$ contains a nonempty open subset of the $(m-2)$-dimensional real hypersurface $\sum_{j=1}^m x_j^2=2\pi^2$ in V. This implies that $\boldsymbol\beta(U)$ is at least $(m-2)$-dimensional.

Since $\boldsymbol\beta$ is real analytic and U is connected, there are only two possibilities:

(a) $\boldsymbol\beta(U)$ is $(m-1)$-dimensional, or

(b) $\boldsymbol\beta(U)$ lies in the real hypersurface $\sum_{j=1}^m x_j^2=2\pi^2$ in V.

However, when $m\geqslant 3$ we can use (27) to eliminate possibility (b). For $\overline{\boldsymbol\beta(U)}$ must contain the limit in (27), which satisfies $\sum_{j=1}^m\beta_j^2=\pi^2\bigl(\frac{1}{k}+\frac{1}{l}\bigr)$. This lies in $\sum_{j=1}^m x_j^2=2\pi^2$ only if $\frac{1}{k}+\frac{1}{l}=2$, that is, if $k=l=1$, since $k,l\geqslant 1$.

Now the ranges of k and l are $k=1,\ldots,a$ and $l=1,\ldots,m-a$. When $m=2$ we are forced to take $k=l=a=1$ and we cannot eliminate possibility (b), as it is actually true. But when $m\geqslant 3$ we are always free to choose $k>1$

or $l > 1$, so $\overline{\boldsymbol{\beta}(U)}$ contains a point not on the hypersurface. Thus (b) is false, so (a) is true, and $\boldsymbol{\beta}(U)$ is $(m-1)$-dimensional. This shows that $\mathrm{d}\boldsymbol{\beta}|_u$ must be surjective at some $u \in U$, and as this is an open condition and $\boldsymbol{\beta}$ is real analytic, $\mathrm{d}\boldsymbol{\beta}|_u$ is surjective for a dense open subset of $u \in U$. $\square$

We immediately deduce the following rough analogue of [7, Cor. 7.14].

Corollary 5.8 *In the situation above, we have $\beta_1, \ldots, \beta_m \in \pi\mathbb{Q}$ for a dense subset of u in U.*

Here is the main result of this section.

Theorem 5.9 *For each $m \geqslant 3$ and $1 \leqslant a < m$, the construction above produces a countably infinite collection of 1-parameter families of distinct special Lagrangian m-folds N in $\mathbb{C}^m$ parametrized by $c \in \mathbb{R}$, given by*

$$\begin{aligned} N = \Big\{ &\big(x_1 \mathrm{e}^{i\theta_1(t)}\sqrt{\alpha_1 + u(t)}, \ldots, x_a \mathrm{e}^{i\theta_a(t)}\sqrt{\alpha_a + u(t)}, \\ &x_{a+1}\mathrm{e}^{i\theta_{a+1}(t)}\sqrt{\alpha_{a+1} - u(t)}, \ldots, x_m \mathrm{e}^{i\theta_m(t)}\sqrt{\alpha_m - u(t)}\ \big) : \\ &t \in \mathbb{R},\ \ x_j \in \mathbb{R},\ \ x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - \cdots - x_m^2 = c\Big\}, \end{aligned} \tag{30}$$

such that

(a) *if $c > 0$ then N is a closed, nonsingular, immersed submanifold diffeomorphic to $\mathcal{S}^{a-1} \times \mathbb{R}^{m-a} \times \mathcal{S}^1$, or to a free quotient of this by $\mathbb{Z}_2$,*

(b) *if $c < 0$ then N is a closed, nonsingular, immersed submanifold diffeomorphic to $\mathbb{R}^a \times \mathcal{S}^{m-a-1} \times \mathcal{S}^1$, or to a free quotient of this by $\mathbb{Z}_2$, and*

(c) *if $c = 0$ then N is a closed, immersed cone with an isolated singular point at 0, diffeomorphic to the cone on $\mathcal{S}^{a-1} \times \mathcal{S}^{m-a-1} \times \mathcal{S}^1$, or to a cone on a free quotient of this by $\mathbb{Z}_2$.*

Proof. We saw at the beginning of §5.5 that if $\beta_1, \ldots, \beta_m \in \pi\mathbb{Q}$ then the time evolution of Theorem 5.2 exists for all t, and is periodic with period nT for some $n \geqslant 1$. But by Corollary 5.8 we have $\beta_1, \ldots, \beta_m \in \pi\mathbb{Q}$ for a dense subset of $u \in U$. Thus, the construction above yields a countable collection of families of SL m-folds, locally parametrized by $\beta_1, \ldots, \beta_m \in \pi\mathbb{Q}$ with $\beta_1 + \cdots + \beta_m = 0$.

Choose one of these families, and define N by (30). Then N is special Lagrangian by Theorem 5.2. Let P be the quadric $x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - \cdots - x_m^2 = c$ in $\mathbb{R}^m$. Then N is the image of a map $\Phi : P \times \mathbb{R} \to \mathbb{C}^m$ taking $\big((x_1, \ldots, x_m), t\big)$ to the point in $\mathbb{C}^m$ defined in (30). As the factors $\mathrm{e}^{i\theta_j(t)}\sqrt{\alpha_j \pm u(t)}$ are always nonzero, Φ is an *immersion* except when $x_1 = \cdots = x_m = 0$, which happens only when $c = 0$.

Thus, N is a nonsingular immersed submanifold when $c \neq 0$, and when $c = 0$ it has just one singular point 0 as an immersed submanifold. Since Φ is periodic in t with period nT, we can instead regard Φ as a map $P \times \mathcal{S}^1 \to \mathbb{C}^m$, where

$\mathcal{S}^1 = \mathbb{R}/nT\,\mathbb{Z}$. It is also not difficult to see that the image N of Φ is *closed*, provided Φ is periodic.

Now P is diffeomorphic to $\mathcal{S}^{a-1}\times\mathbb{R}^{m-a}$ when $c>0$, to $\mathbb{R}^a\times\mathcal{S}^{m-a-1}$ when $c<0$, and to the cone on $\mathcal{S}^{a-1}\times\mathcal{S}^{m-a-1}$ when $c=0$. Thus N is diffeomorphic under Φ as an immersed submanifold to $\mathcal{S}^{a-1}\times\mathbb{R}^{m-a}\times\mathcal{S}^1$ when $c>0$, to $\mathbb{R}^a\times\mathcal{S}^{m-a-1}\times\mathcal{S}^1$ when $c<0$, and to the cone on $\mathcal{S}^{a-1}\times\mathcal{S}^{m-a-1}\times\mathcal{S}^1$ when $c=0$.

It remains only to discuss the parts about free quotients by $\mathbb{Z}_2$ in (a)–(c). We could have left these bits out, as the result is true without them. The point is this: suppose $\beta_j=\pi a_j/b$, for integers $a_1,\ldots,a_m$ and b with $\mathrm{hcf}(a_1,\ldots,a_m,b)=1$ and $b>0$. Then $w_j(t+bT)=(-1)^{a_j}w_j(t)$, so that w_j has period bT if a_j is even, and $2bT$ if a_j is odd. Thus Φ satisfies

$$\Phi\bigl((x_1,\ldots,x_m),t\bigr)=\Phi\bigl(\bigl((-1)^{a_1}x_1,\ldots,(-1)^{a_m}x_m\bigr),t+bT\bigr).$$

Since P is invariant under $(x_1,\ldots,x_m)\mapsto\bigl((-1)^{a_1}x_1,\ldots,(-1)^{a_m}x_m\bigr)$, the family of quadrics making up N has period bT. But it doesn't simply repeat after time bT, but also changes the signs of those x_j with a_j odd. Let us regard Φ as mapping $P\times\mathcal{S}^1\rightarrow\mathbb{C}^m$, where $\mathcal{S}^1=\mathbb{R}/2bT\,\mathbb{Z}$. Then Φ is generically 2:1, and filters through a map $(P\times\mathcal{S}^1)/\mathbb{Z}_2\rightarrow\mathbb{C}^m$, where the generator of $\mathbb{Z}_2$ acts freely on $P\times\mathcal{S}^1$ by

$$\bigl((x_1,\ldots,x_m),t+2bT\,\mathbb{Z}\bigr)\mapsto\bigl(\bigl((-1)^{a_1}x_1,\ldots,(-1)^{a_n}x_m\bigr),t+bT+2bT\,\mathbb{Z}\bigr).$$

This completes the proof. $\square$

This theorem is analogous to [7, Th. 7.15]. However, [7, Th. 7.15] constructs SL T^{m-1}-cones N with rather large symmetry groups $\mathrm{Sym}^0(N)=\mathrm{U}(1)^{m-2}$, but the most generic cones and more general SL m-folds constructed above have $\mathrm{Sym}^0(N)=\{1\}$, so they have rather small symmetry groups. This is not true for all the m-folds of Theorem 5.9, but only when $\alpha_1,\ldots,\alpha_a$ and $\alpha_{a+1},\ldots,\alpha_m$ are distinct.

Part (c) of the theorem is interesting, as it provides a large family of singular special Lagrangian cones in $\mathbb{C}^m$ which are good local models for the singularities of special Lagrangian m-folds in Calabi–Yau m-folds. Parts (a) and (b) are examples of *Asymptotically Conical* special Lagrangian m-folds in $\mathbb{C}^m$, and also give local models for how the singularities of part (c) can appear as limits of families of nonsingular SL m-folds in Calabi–Yau m-folds.

Here is a crude 'parameter count' of the number of distinct families of special Lagrangian m-folds produced by this construction. Locally the families are parametrized by by $\beta_1,\ldots,\beta_m\in\pi\mathbb{Q}$ with $\beta_1+\cdots+\beta_m=0$. There are unique integers $a_1,\ldots,a_m,b$ with $\beta_j=\pi a_j/b$, such that $\mathrm{hcf}(a_1,\ldots,a_m,b)=1$ and $b>0$. But $a_1+\cdots+a_m=0$, so we can discard a_m.

Observe that the constructions of this section and of [7, §7] are strikingly similar in some ways, despite their differences. The o.d.e.s (9) and [7, eq. (8)] behind the two constructions are essentially the same. And although the periodicity conditions considered above and in [7, §7.5] are very different, the end results are similar, as above we saw that N depends on m integers $a_1,\ldots,a_{m-1},b$

with highest common factor 1, whereas after [7, Th. 7.15] we concluded that N depended on m integers $\tilde{a}_1, \ldots, \tilde{a}_{m-1}, a$ with highest common factor 1.

The author wonders whether there is some deep connection, or duality, between the constructions of [7, §7] and this section, which explains these similarities. This could be an integrable systems phenomenon, some kind of 'Bäcklund transformation' between the two constructions which respects the periodicity criteria, or something to do with mirror symmetry.

6 The 3-dimensional case

We now specialize to the case $m = 3$ in the situation of §5. The special Lagrangian 3-folds we discuss in this section were also considered from a different point of view by Bryant [1, §3.5]. Bryant uses Cartan–Kähler theory to study special Lagrangian 3-folds L in $\mathbb{C}^3$ whose second fundamental form h satisfies certain conditions at every point.

In effect, Bryant shows [1, Th. 4] that L is one of the SL 3-folds of parts (b)–(d) of §5.4 if and only if h has stabilizer $\mathbb{Z}_2$ in a dense open subset of L. His methods are local. They show that the family of such 3-folds is finite-dimensional and compute the dimension, but give less information on the global nature of L.

Of the four cases (a)–(d) in §5.4, cases (a), (b) and (c) are already well understood, so we will concentrate on case (d). Fixing $m = 3$, the two possibilities $a = 1$ and $a = 2$ in this case are exchanged by reversing the order of z_1, z_2, z_3 and changing the sign of c, so without loss of generality we shall choose $a = 1$.

We begin by summarizing the results of §5.1 and §5.2 when $m = 3$ and $a = 1$. From Theorem 5.1 we obtain

Theorem 6.1 *Suppose $w_1, w_2, w_3 : (-\epsilon, \epsilon) \to \mathbb{C} \setminus \{0\}$ satisfy*

$$\frac{\mathrm{d}w_1}{\mathrm{d}t} = \overline{w_2 w_3}, \quad \frac{\mathrm{d}w_2}{\mathrm{d}t} = -\,\overline{w_3 w_1} \quad \textit{and} \quad \frac{\mathrm{d}w_3}{\mathrm{d}t} = -\,\overline{w_1 w_2}. \tag{31}$$

Let $c \in \mathbb{R}$, and define a subset N of $\mathbb{C}^3$ to be

$$\Big\{\big(w_1(t)x_1, w_2(t)x_2, w_3(t)x_3\big) : t \in (-\epsilon, \epsilon), \;\; x_j \in \mathbb{R}, \;\; x_1^2 - x_2^2 - x_3^2 = c\Big\}. \tag{32}$$

Then N is a special Lagrangian submanifold in $\mathbb{C}^3$.

Combining Theorem 5.2, Proposition 5.3 and ideas from §5.4, we get

Proposition 6.2 *In the situation of Theorem 6.1 the functions w_1, w_2, w_3 may be written*

$$w_1 = \mathrm{e}^{i\theta_1}\sqrt{\alpha_1 + u}, \quad w_2 = \mathrm{e}^{i\theta_2}\sqrt{\alpha_2 - u} \quad \textit{and} \quad w_3 = \mathrm{e}^{i\theta_3}\sqrt{\alpha_3 - u},$$

so that

$$|w_1|^2 = \alpha_1 + u, \quad |w_2|^2 = \alpha_2 - u \quad \textit{and} \quad |w_3|^2 = \alpha_3 - u, \tag{33}$$

where $\alpha_j \in \mathbb{R}$ and $u, \theta_1, \theta_2, \theta_3 : (-\epsilon, \epsilon) \to \mathbb{R}$ are differentiable functions. Define

$$Q(u) = (\alpha_1 + u)(\alpha_2 - u)(\alpha_3 - u) \quad \textit{and} \quad \theta = \theta_1 + \theta_2 + \theta_3.$$

Then $Q(u)^{1/2} \sin\theta \equiv A$ for some $A \in \mathbb{R}$, and u and θ_j satisfy

$$\begin{aligned} \Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)^2 &= 4\big(Q(u) - A^2\big), & \frac{\mathrm{d}\theta_1}{\mathrm{d}t} &= -\frac{A}{\alpha_1 + u}, \\ \frac{\mathrm{d}\theta_2}{\mathrm{d}t} &= \frac{A}{\alpha_2 - u} \qquad \textit{and} & \frac{\mathrm{d}\theta_3}{\mathrm{d}t} &= \frac{A}{\alpha_3 - u}. \end{aligned} \tag{34}$$

If $A \neq 0$ then w_j, u and θ_j exist for all t in $\mathbb{R}$, not just in $(-\epsilon, \epsilon)$.

In the last line, the w_j actually exist for all t even when $A = 0$. But in this case at least one of the w_j will become zero at some time t, and then θ_j is undefined at time t, and should be regarded as jumping discontinuously by $\pm\pi$. Note that as $m = 3$, by following the method of [7, §8.2] we can solve equation (34) explicitly using the Jacobi elliptic functions. But we will not do this here.

We will explain the 3-dimensional analogue of Theorem 5.9 in a little more detail. From §5.5, when $m = 3$ and $a = 1$ the SL 3-folds of Theorem 5.9 are locally parametrized by $\beta_1, \beta_2, \beta_3 \in \pi\mathbb{Q}$ with $\beta_1 < 0$, $\beta_2, \beta_3 > 0$ and $\beta_1 + \beta_2 + \beta_3 = 0$. We may write such β_j uniquely as $\beta_j = \pi a_j / b$, where $a_1, a_2, a_3, b \in \mathbb{Z}$ with $b > 0$ and $\mathrm{hcf}(a_1, a_2, a_3, b) = 1$. Then the family of quadrics making up N has period bT.

However, the functions $w_1, w_2, w_3 : \mathbb{R} \to \mathbb{C}$ satisfy $w_j(t + bT) = (-1)^{a_j} w_j(t)$ for $t \in \mathbb{R}$. Thus, if a_j is odd then w_j actually has period $2bT$ rather than bT. The family of quadrics making up N still has period bT, because the quadric $x_1^2 - x_2^2 - x_3^2 = c$ in $\mathbb{R}^3$ is invariant under a change of sign of x_j.

Now in describing the topological type of N in parts (a)–(c) of Theorem 5.9, we allowed the possibility of a free quotient by $\mathbb{Z}_2$. When $m = 3$ and $a = 1$, how this $\mathbb{Z}_2$ acts depends on whether a_1 is even or odd. For instance, when $c > 0$ the quadric $x_1^2 - x_2^2 - x_3^2 = c$ splits into two connected components, with $x_1 > 0$ and $x_1 < 0$. Replacing t by $t + bT$ maps x_j to $(-1)^{a_j} x_j$. When a_1 is even this map fixes the two components of the quadric, so that N splits into two pieces, but when a_1 is odd the two components are swapped, so that N comes in only one piece.

We shall state two versions of Theorem 5.9 when $m = 3$ and $a = 1$, for the two cases a_1 even and a_1 odd. Note that the sets of triples $(\beta_1, \beta_2, \beta_3)$ with $\beta_j \in \pi\mathbb{Q}$ and a_1 even, and with a_1 odd, are both dense in the set of all $(\beta_1, \beta_2, \beta_3)$, so by the argument of Corollary 5.8 the sets of initial data with $\beta_j \in \pi\mathbb{Q}$ and a_1 even, and with $\beta_j \in \pi\mathbb{Q}$ and a_1 odd, are both dense in the set of all initial data, and for both cases there are a countably infinite number of solutions.

Here is the first version, with a_1 even.

Theorem 6.3 *The construction above gives a countably infinite collection of* 1-*parameter families of distinct special Lagrangian* 3-*folds N in $\mathbb{C}^3$ parametrized*

by $c \in \mathbb{R}$, given by

$$N = \Big\{ \big(x_1 \mathrm{e}^{i\theta_1(t)} \sqrt{\alpha_1 + u(t)}, x_2 \mathrm{e}^{i\theta_2(t)} \sqrt{\alpha_2 - u(t)}, x_3 \mathrm{e}^{i\theta_3(t)} \sqrt{\alpha_3 - u(t)} \,\big) : \\ t \in \mathbb{R}, \quad x_j \in \mathbb{R}, \quad x_1^2 - x_2^2 - x_3^2 = c \Big\},$$

such that

(a) *if $c > 0$ then N is the union of two distinct pieces N_+ and $N_- = -N_+$, each of which is a closed, nonsingular, immersed submanifold diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$.*

(b) *if $c < 0$ then N is a closed, nonsingular, immersed submanifold diffeomorphic to $T^2 \times \mathbb{R}$, with $N = -N$, and*

(c) *if $c = 0$ then N is the union of two distinct pieces N_+ and $N_- = -N_+$, each of which is a closed, immersed cone on T^2, with an isolated singular point at 0.*

Here is the second version, with a_1 odd.

Theorem 6.4 *The construction above gives a countably infinite collection of 1-parameter families of distinct special Lagrangian 3-folds N in $\mathbb{C}^3$ parametrized by $c \in \mathbb{R}$, given by*

$$N = \Big\{ \big(x_1 \mathrm{e}^{i\theta_1(t)} \sqrt{\alpha_1 + u(t)}, x_2 \mathrm{e}^{i\theta_2(t)} \sqrt{\alpha_2 - u(t)}, x_3 \mathrm{e}^{i\theta_3(t)} \sqrt{\alpha_3 - u(t)} \,\big) : \\ t \in \mathbb{R}, \quad x_j \in \mathbb{R}, \quad x_1^2 - x_2^2 - x_3^2 = c \Big\},$$

such that

(a) *if $c > 0$ then N is a closed, nonsingular, immersed submanifold diffeomorphic to $\mathcal{S}^1 \times \mathbb{R}^2$.*

(b) *if $c < 0$ then N is a closed, nonsingular, immersed submanifold diffeomorphic to a free quotient of $T^2 \times \mathbb{R}$ by $\mathbb{Z}_2$. It can be thought of as the total space of a nontrivial real line bundle over the Klein bottle, and has only one infinite end, diffeomorphic to $T^2 \times (0, \infty)$.*

(c) *if $c = 0$ then N is a closed, immersed cone on T^2, with an isolated singular point at 0.*

In all three cases we have $N = -N$.

In part (c) of these two theorems, the author expects the T^2-cones to be *embedded* in nearly all cases.

6.1 Conformal parametrization of SL cones

Let us now put $c = 0$ in Theorem 6.1, so that the 3-fold N of (32) is a cone. Define $\Sigma = N \cap \mathcal{S}^5$, where $\mathcal{S}^5$ is the unit sphere in $\mathbb{C}^3$. Then Σ is a *minimal Legendrian surface* in $\mathcal{S}^5$, as N is a minimal Lagrangian 3-fold in $\mathbb{C}^3$.

We shall write down an explicit *conformal parametrization* $\Phi : \mathbb{R}^2 \to \Sigma$. Now by [3, p. 32], a conformal map from a Riemann surface to a Riemannian manifold is harmonic if and only if its image in minimal. Thus, as Φ is conformal and its image Σ is minimal, Φ is harmonic; and so we have constructed an *explicit harmonic map* $\Phi : \mathbb{R}^2 \to \mathcal{S}^5$. Such maps are of interest to people who study harmonic maps and integrable systems. We begin with a preliminary lemma.

Lemma 6.5 *Let w_j be as in Theorem 6.1 and α_j and u be as in Proposition 6.2. Then $\Sigma = N \cap \mathcal{S}^5$ may be written*

$$\Big\{\big(w_1(t)x_1, w_2(t)x_2, w_3(t)x_3\big) : t \in \mathbb{R}, \quad x_j \in \mathbb{R}, \\ \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 = 1, \quad x_1^2 - x_2^2 - x_3^2 = 0\Big\}.$$

Proof. A point (w_1x_1, w_2x_2, w_3x_3) in N lies in Σ if and only if $|w_1|^2x_1^2 + |w_2|^2x_2^2 + |w_3|^2x_3^2 = 1$. Substituting in (33), this is equivalent to

$$(\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2) + (x_1^2 - x_2^2 - x_3^2) = 1.$$

But by definition $x_1^2 - x_2^2 - x_3^2 = 0$, and thus $\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 = 1$. □

This shows that Σ is naturally isomorphic to $C \times \mathbb{R}$, where C is given by

$$C = \big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 = 1,\ x_1^2 - x_2^2 - x_3^2 = 0\big\}.$$

Since we may assume as in §5.4 that $\alpha_j > 0$ for $j = 1, 2, 3$, it follows that C divides into two connected components C_+, with $x_1 > 0$, and C_-, with $x_1 < 0$, each of which is diffeomorphic to $\mathcal{S}^1$. This splitting into $C_\pm$ corresponds to the splitting of N into $N_\pm$ in part (c) of Theorem 6.4. There is also a corresponding splitting of Σ into $\Sigma_\pm$.

Let us parametrize the circle C_+ with a parameter s, so that

$$C_+ = \big\{\big(x_1(s), x_2(s), x_3(s)\big) : s \in \mathbb{R}\big\}.$$

This gives a parametrization $\Phi : \mathbb{R}^2 \to \Sigma_+$ of Σ_+, by

$$\Phi : (s, t) \mapsto \big(w_1(t)x_1(s), w_2(t)x_2(s), w_3(t)x_3(s)\big). \tag{35}$$

We shall calculate the conditions upon $x_j(s)$ for Φ to be conformal, and solve them.

Since the $x_j(s)$ satisfy $\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 = 1$ and $x_1^2 - x_2^2 - x_3^2 = 0$, differentiating with respect to s gives

$$\alpha_1 x_1 \dot{x}_1 + \alpha_2 x_2 \dot{x}_2 + \alpha_3 x_3 \dot{x}_3 = 0 \quad \text{and} \quad x_1 \dot{x}_1 - x_2 \dot{x}_2 - x_3 \dot{x}_3 = 0,$$

where '$\dot{\ }$' is $\frac{\mathrm{d}}{\mathrm{d}s}$. Thus the vector $(\dot{x}_1, \dot{x}_2, \dot{x}_3)$ is orthogonal to $(\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3)$ and $(x_1, -x_2, -x_3)$, so it is parallel to their vector product. This gives

$$\begin{gathered} \dot{x}_1 = \gamma(\alpha_2 - \alpha_3)x_2 x_3, \qquad \dot{x}_2 = -\gamma(\alpha_1 + \alpha_3)x_3 x_1 \\ \text{and} \qquad \dot{x}_3 = \gamma(\alpha_1 + \alpha_2)x_1 x_2, \end{gathered} \tag{36}$$

for some real nonzero function $\gamma(s)$. Also, as x_1, x_2, x_3 satisfy $\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 = 1$ and $x_1^2 - x_2^2 - x_3^2 = 0$, we may write

$$x_1^2 = \frac{1 + (\alpha_2 - \alpha_3)v}{\alpha_1 + \alpha_2}, \quad x_2^2 = \frac{1 - (\alpha_1 + \alpha_3)v}{\alpha_1 + \alpha_2} \quad \text{and} \quad x_3^2 = v, \tag{37}$$

for some real function $v(s)$.

Combining equations (31), (35) and (36) gives

$$\begin{aligned} \frac{\partial \Phi}{\partial s} &= \gamma\big((\alpha_2 - \alpha_3)w_1 x_2 x_3, -(\alpha_1 + \alpha_3)w_2 x_3 x_1, (\alpha_1 + \alpha_2)w_3 x_1 x_2\big) \\ \text{and} \quad \frac{\partial \Phi}{\partial t} &= \big(\overline{w_2 w_3}\, x_1, -\overline{w_3 w_1}\, x_2, -\overline{w_1 w_2}\, x_3\big). \end{aligned}$$

Thus

$$g\Big(\frac{\partial \Phi}{\partial s}, \frac{\partial \Phi}{\partial t}\Big) = \gamma\big((\alpha_2 - \alpha_3) + (\alpha_1 + \alpha_3) - (\alpha_1 + \alpha_2)\big)\,\mathrm{Re}(w_1 w_2 w_3)x_1 x_2 x_3 = 0,$$

so that $\frac{\partial \Phi}{\partial s}$ and $\frac{\partial \Phi}{\partial t}$ are orthogonal.

Using equations (33) and (37) to write $\big|\frac{\partial \Phi}{\partial s}\big|^2$ and $\big|\frac{\partial \Phi}{\partial t}\big|^2$ in terms of u and v, after a lot of cancellation we find that

$$\begin{aligned} \Big|\frac{\partial \Phi}{\partial s}\Big|^2 &= \gamma^2\big(\alpha_3 + u + (\alpha_2 - \alpha_3)(\alpha_1 + \alpha_3)v\big) \\ \text{and} \quad \Big|\frac{\partial \Phi}{\partial t}\Big|^2 &= \alpha_3 + u + (\alpha_2 - \alpha_3)(\alpha_1 + \alpha_3)v. \end{aligned}$$

Note that the coefficients of uv, v^2 and uv^2 in $\big|\frac{\partial \Phi}{\partial s}\big|^2$ and the coefficients of uv, u^2 and $u^2 v$ in $\big|\frac{\partial \Phi}{\partial t}\big|^2$ all vanish. From these equations, we see that if $\gamma^2 = 1$ then $\big|\frac{\partial \Phi}{\partial s}\big|^2 = \big|\frac{\partial \Phi}{\partial t}\big|^2$, so that Φ is *conformal*.

So let us fix $\gamma = 1$. Then we seek functions $x_1(s), x_2(s), x_3(s)$ satisfying the o.d.e. (36) with $\gamma = 1$, and the restrictions (37). It turns out that we can solve these equations explicitly in terms of the *Jacobi elliptic functions*, to which we now give a brief introduction. The following material can be found in Chandrasekharan [2, Ch. VII].

For each $k \in [0, 1]$, the Jacobi elliptic functions $\mathrm{sn}(t, k)$, $\mathrm{cn}(t, k)$, $\mathrm{dn}(t, k)$ with modulus k are the unique solutions to the o.d.e.s

$$\begin{aligned} \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{sn}(t,k)\big)^2 &= \big(1 - \mathrm{sn}^2(t,k)\big)\big(1 - k^2\mathrm{sn}^2(t,k)\big), \\ \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{cn}(t,k)\big)^2 &= \big(1 - \mathrm{cn}^2(t,k)\big)\big(1 - k^2 + k^2\mathrm{cn}^2(t,k)\big), \\ \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{dn}(t,k)\big)^2 &= -\big(1 - \mathrm{dn}^2(t,k)\big)\big(1 - k^2 - \mathrm{dn}^2(t,k)\big), \end{aligned}$$

with initial conditions

$$\begin{aligned}
\mathrm{sn}(0,k)=0, \quad \mathrm{cn}(0,k)=1, \quad \mathrm{dn}(0,k)=1,\\
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{sn}(0,k)=1,\ \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{cn}(0,k)=0,\ \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{dn}(0,k)=0.
\end{aligned}$$

They satisfy the identities

$$\mathrm{sn}^2(t,k)+\mathrm{cn}^2(t,k)=1 \ \text{ and } \ k^2\mathrm{sn}^2(t,k)+\mathrm{dn}^2(t,k)=1, \tag{38}$$

and the differential equations

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{sn}(t,k)=\mathrm{cn}(t,k)\mathrm{dn}(t,k), \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{cn}(t,k)=-\mathrm{sn}(t,k)\mathrm{dn}(t,k)\\
\text{and} \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{dn}(t,k)=-k^2\mathrm{sn}(t,k)\mathrm{cn}(t,k).
\end{aligned} \tag{39}$$

Returning to equations (36) and (37), suppose $\alpha_2 \leqslant \alpha_3$, and define

$$\begin{aligned}
x_1=(\alpha_1+\alpha_2)^{-1/2}\mathrm{dn}(\mu s,\nu), \quad x_2=(\alpha_1+\alpha_2)^{-1/2}\mathrm{cn}(\mu s,\nu)\\
\text{and} \qquad x_3=(\alpha_1+\alpha_3)^{-1/2}\mathrm{sn}(\mu s,\nu),
\end{aligned}$$

where

$$\mu=(\alpha_1+\alpha_3)^{1/2} \quad \text{and} \quad \nu^2=\frac{\alpha_3-\alpha_2}{\alpha_1+\alpha_3}.$$

Then from (38) and (39), these x_j satisfy (36) and (37) with $v=(\alpha_1+\alpha_3)^{-1}\mathrm{sn}^2(\mu s,\nu)$. Drawing the above work together, we have proved:

Theorem 6.6 *In the situation above, define $\Phi:\mathbb{R}^2\to\mathcal{S}^5$ by*

$$\begin{aligned}
\Phi:(s,t)\mapsto \big((\alpha_1+\alpha_2)^{-1/2}\mathrm{dn}(\mu s,\nu)w_1(t),\\
(\alpha_1+\alpha_2)^{-1/2}\mathrm{cn}(\mu s,\nu)w_2(t),(\alpha_1+\alpha_3)^{-1/2}\mathrm{sn}(\mu s,\nu)w_3(t)\big),
\end{aligned} \tag{40}$$

where $\mu=(\alpha_1+\alpha_3)^{1/2}$, $\nu=(\alpha_3-\alpha_2)^{1/2}(\alpha_1+\alpha_3)^{-1/2}$ and $\mathcal{S}^5$ is the unit sphere in $\mathbb{C}^3$. Then Φ is a conformal, harmonic map.

We made the assumption above that $\alpha_2 \leqslant \alpha_3$. If $\alpha_2 > \alpha_3$ then we can apply the same method, but swapping over x_2 and x_3, and α_2 and α_3, so that

$$\begin{aligned}
x_1=(\alpha_1+\alpha_3)^{-1/2}\mathrm{dn}(\mu s,\nu), \quad x_2=(\alpha_1+\alpha_2)^{-1/2}\mathrm{sn}(\mu s,\nu)\\
\text{and} \qquad x_3=(\alpha_1+\alpha_3)^{-1/2}\mathrm{cn}(\mu s,\nu),
\end{aligned}$$

where

$$\mu=(\alpha_1+\alpha_2)^{1/2} \quad \text{and} \quad \nu^2=\frac{\alpha_2-\alpha_3}{\alpha_1+\alpha_2}.$$

Note also that all of our expressions for $x_j(s)$ depend only on the linear combinations $\alpha_1+\alpha_2$, $\alpha_1+\alpha_3$ and $\alpha_2-\alpha_3$ of $\alpha_1,\alpha_2,\alpha_3$. This is because the α_j were defined in (12) up to an arbitrary constant λ, and these combinations are independent of λ.

6.2 Relation with harmonic tori in $\mathbb{CP}^2$ and $\mathcal{S}^5$

Theorem 6.6 constructed a family of *explicit conformal harmonic maps* $\Phi : \mathbb{R}^2 \to \mathcal{S}^5$. Furthermore, as the cone on the image of Φ is Lagrangian, one can show that if $\pi : \mathcal{S}^5 \to \mathbb{CP}^2$ is the Hopf projection then $\pi \circ \Phi$ is conformal and harmonic, so we also have a family of explicit conformal harmonic maps $\Psi : \mathbb{R}^2 \to \mathbb{CP}^2$.

Now harmonic maps from Riemann surfaces into spheres and projective spaces are an *integrable system*, and have been intensively studied in the integrable systems literature. For an introduction to the subject, see Fordy and Wood [3], in particular the articles by Bolton and Woodward [3, p. 59–82], McIntosh [3, p. 205–220] and Burstall and Pedit [3, p. 221–272].

Therefore our examples can be analyzed from the integrable systems point of view. We postpone this analysis to the sequel [11]. In [11, §5] we shall realize the SL cones in $\mathbb{C}^3$ constructed in Theorem 6.1 with $c = 0$ as special cases of a more general construction of special Lagrangian cones in $\mathbb{C}^3$, which involves two commuting o.d.e.s.

Then in [11, §6] we work through the integrable systems framework for the corresponding family of harmonic maps $\Psi : \mathbb{R}^2 \to \mathbb{CP}^2$, showing that they are generically superconformal of finite type, and determining their harmonic sequences, Toda solutions, algebras of polynomial Killing fields, and spectral curves. From the integrable systems point of view, part (c) of Theorems 6.3 and 6.4 are interesting because they construct large families of *superconformal harmonic tori* in $\mathbb{CP}^2$.

7 Examples from evolving non-centred quadrics

We will now apply the construction of §3 to the family of sets of affine evolution data (P, χ) defined using non-centred quadrics in $\mathbb{R}^m$ in Example 4.4 of §4. Our treatment follows §5 closely, and so we will leave out many of the details.

As in Example 4.4, let $(m-1)/2 \leqslant a \leqslant m-1$, and define P and χ by

$$P = \bigl\{(x_1, \dots, x_m) \in \mathbb{R}^m : x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - \cdots - x_{m-1}^2 + 2x_m = 0\bigr\},$$

$$\begin{aligned}\chi = 2(-1)^{m-1} e_1 \wedge \cdots \wedge e_{m-1} + 2\sum_{j=1}^{a} (-1)^{j-1} x_j e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_m \\ - 2\sum_{j=a+1}^{m-1} (-1)^{j-1} x_j\, e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_m,\end{aligned}$$

where e_j is the vector with $x_j = 1$ and $x_k = 0$ for $j \neq k$. Then P is nonsingular in $\mathbb{R}^m$, and (P, χ) is a set of *affine evolution data*.

Consider affine maps $\phi : \mathbb{R}^m \to \mathbb{C}^m$ of the form

$$\phi : (x_1, \dots, x_m) \mapsto (w_1 x_1, \dots, w_{m-1} x_{m-1}, x_m + \beta) \tag{41}$$

for $w_1, \dots, w_{m-1}$ in $\mathbb{C} \setminus \{0\}$ and $\beta \in \mathbb{C}$. Then ϕ is injective and $\operatorname{Im} \phi$ is an affine Lagrangian m-plane in $\mathbb{C}^m$, so that ϕ lies in the subset $\mathcal{C}_P$ of $\operatorname{Aff}(\mathbb{R}^m, \mathbb{C}^m)$ given in Definition 3.4.

Then as in §5, the evolution equation (2) for ϕ in $\mathcal{C}_P$ preserves ϕ of the form (41). So, consider a 1-parameter family $\{\phi_t : t \in (-\epsilon, \epsilon)\}$ given by

$$\phi_t : (x_1, \ldots, x_m) \mapsto \big(w_1(t)x_1, \ldots, w_{m-1}(t)x_{m-1}, x_m + \beta(t)\big),$$

where $w_1, \ldots, w_{m-1} : (-\epsilon, \epsilon) \to \mathbb{C} \setminus \{0\}$ and $\beta : (-\epsilon, \epsilon) \to \mathbb{C}$ are differentiable functions. Following the method of §5.1 one can rewrite (2) as a first-order o.d.e. upon $w_1, \ldots, w_{m-1}$ and β. We end up with the following analogue of Theorem 5.1.

Theorem 7.1 *Let $(m-1)/2 \leqslant a \leqslant m-1$. Suppose $w_1, \ldots, w_{m-1} : (-\epsilon, \epsilon) \to \mathbb{C} \setminus \{0\}$ and $\beta : (-\epsilon, \epsilon) \to \mathbb{C} \setminus \{0\}$ are differentiable functions satisfying*

$$\frac{\mathrm{d}w_j}{\mathrm{d}t} = \begin{cases} \overline{w_1 \cdots w_{j-1} w_{j+1} \cdots w_{m-1}}, & j = 1, \ldots, a, \\ -\overline{w_1 \cdots w_{j-1} w_{j+1} \cdots w_{m-1}}, & j = a+1, \ldots, m-1, \end{cases} \tag{42}$$

$$\textit{and} \qquad \frac{\mathrm{d}\beta}{\mathrm{d}t} = \overline{w_1 \cdots w_{m-1}}. \tag{43}$$

Define a subset N of $\mathbb{C}^m$ by

$$\begin{aligned} N = \Big\{ \big(w_1(t)x_1, \ldots, w_{m-1}(t)x_{m-1}, x_m + \beta(t)\big) : t \in (-\epsilon, \epsilon), \\ x_j \in \mathbb{R}, \quad x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - \cdots - x_{m-1}^2 + 2x_m = 0 \Big\}. \end{aligned} \tag{44}$$

Then N is a special Lagrangian submanifold in $\mathbb{C}^m$.

Now (42) shows that the evolution of $w_1, \ldots, w_{m-1}$ is independent of β. Furthermore, equation (42) coincides with equation (9) of Theorem 5.1, with m replaced by $m-1$. Thus, we can use the material of §5.2–§5.5 to write $w_1, \ldots, w_{m-1}$ explicitly in terms of elliptic integrals, and to describe their global behaviour.

Having found $w_1, \ldots, w_{m-1}$ as functions of t, we can then use (43) to determine the function β. Thus we can solve equations (42) and (43) in a fairly explicit way, and use the solution to describe and understand the SL m-fold N of (44).

So, following §5.2, let $\lambda \in \mathbb{R}$, set $\alpha_j = |w_j(0)|^2 - \lambda$ for $j = 1, \ldots, a$ and $\alpha_j = |w_j(0)|^2 + \lambda$ for $j = a+1, \ldots, m-1$, and define $u : (-\epsilon, \epsilon) \to \mathbb{R}$ by $u(t) = \lambda + 2\int_0^t \mathrm{Re}\big(w_1(s) \cdots w_{m-1}(s)\big)\mathrm{d}s$. Then we have

$$w_j(t) = \begin{cases} \mathrm{e}^{i\theta_j(t)}\sqrt{\alpha_j + u(t)}, & j = 1, \ldots, a, \\ \mathrm{e}^{i\theta_j(t)}\sqrt{\alpha_j - u(t)}, & j = a+1, \ldots, m-1, \end{cases}$$

for differentiable functions $\theta_1, \ldots, \theta_{m-1} : (-\epsilon, \epsilon) \to \mathbb{R}$. Define

$$\theta = \theta_1 + \cdots + \theta_{m-1} \quad \text{and} \quad Q(u) = \prod_{j=1}^{a}(\alpha_j + u) \prod_{j=a+1}^{m-1}(\alpha_j - u).$$

Then following (14)–(16) we find that $\frac{\mathrm{d}u}{\mathrm{d}t}=2Q(u)^{1/2}\cos\theta$, and derive expressions for $\frac{\mathrm{d}\theta_j}{\mathrm{d}t}$ and $\frac{\mathrm{d}\theta}{\mathrm{d}t}$.

As in (18) we show that $Q(u)^{1/2}\sin\theta\equiv A$ for some constant $A\in\mathbb{R}$. Now $w_1\dots w_{m-1}=Q(u)^{1/2}\mathrm{e}^{i\theta}$. Thus equation (43) gives

$$\frac{\mathrm{d}\beta}{\mathrm{d}t}=Q(u)^{1/2}(\cos\theta-i\sin\theta)=\tfrac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}t}-iA,$$

as $Q(u)^{1/2}\cos\theta=\frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}t}$ and $Q(u)^{1/2}\sin\theta=A$. Integrating this gives

$$\beta(t)=C+\tfrac{1}{2}u(t)-iAt, \tag{45}$$

where $C=\beta(0)-\frac{1}{2}u(0)$. As $\beta(0)$ is arbitrary we may as well fix $C=0$. So we obtain the following analogue of Theorem 5.2.

Theorem 7.2 *Let u and $\theta_1,\dots,\theta_{m-1}$ be differentiable functions $(-\epsilon,\epsilon)\to\mathbb{R}$ satisfying*

$$\frac{\mathrm{d}u}{\mathrm{d}t}=2Q(u)^{1/2}\cos\theta$$
$$\text{and}\quad \frac{\mathrm{d}\theta_j}{\mathrm{d}t}=\begin{cases}-\dfrac{Q(u)^{1/2}\sin\theta}{\alpha_j+u}, & j=1,\dots,a,\\[2mm] \dfrac{Q(u)^{1/2}\sin\theta}{\alpha_j-u}, & j=a+1,\dots,m-1,\end{cases}$$

where $\theta=\theta_1+\cdots+\theta_{m-1}$, so that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=-Q(u)^{1/2}\sin\theta\left(\sum_{j=1}^{a}\frac{1}{\alpha_j+u}-\sum_{j=a+1}^{m-1}\frac{1}{\alpha_j-u}\right).$$

Then u and θ satisfy $Q(u)^{1/2}\sin\theta\equiv A$ for some $A\in\mathbb{R}$. Suppose that $\alpha_j+u>0$ for $j=1,\dots,a$ and $\alpha_j-u>0$ for $j=a+1,\dots,m-1$ and $t\in(-\epsilon,\epsilon)$. Define a subset N of $\mathbb{C}^m$ to be

$$\Bigl\{\bigl(x_1\mathrm{e}^{i\theta_1(t)}\sqrt{\alpha_1+u(t)},\dots,x_a\mathrm{e}^{i\theta_a(t)}\sqrt{\alpha_a+u(t)},x_{a+1}\mathrm{e}^{i\theta_{a+1}(t)}\sqrt{\alpha_{a+1}-u(t)},$$
$$\dots,x_{m-1}\mathrm{e}^{i\theta_{m-1}(t)}\sqrt{\alpha_{m-1}-u(t)},x_m+\tfrac{1}{2}u(t)-iAt\bigr):$$
$$t\in(-\epsilon,\epsilon),\;\; x_j\in\mathbb{R},\;\; x_1^2+\cdots+x_a^2-x_{a+1}^2-\cdots-x_{m-1}^2+2x_m=0\Bigr\}.$$

Then N is a special Lagrangian submanifold in $\mathbb{C}^m$.

As in §5.3, if we assume that $\theta(t)\in(-\pi/2,\pi/2)$ for $t\in(-\epsilon,\epsilon)$ then u is an increasing function of t, and we can choose to regard everything as a function of u rather than of t. This yields the following analogue of Theorem 5.4:

Theorem 7.3 *Suppose $\theta(t) \in (-\pi/2, \pi/2)$ for all $t \in (-\epsilon, \epsilon)$. Then the special Lagrangian m-fold N of Theorem 7.1 is given explicitly by*

$$\begin{aligned} N = \Big\{ \Big(& x_1 \mathrm{e}^{i\theta_1(u)} \sqrt{\alpha_1 + u}, \dots, x_a \mathrm{e}^{i\theta_a(u)} \sqrt{\alpha_a + u}, x_{a+1} \mathrm{e}^{i\theta_{a+1}(u)} \sqrt{\alpha_{a+1} - u}, \\ & \dots, x_{m-1} \mathrm{e}^{i\theta_{m-1}(u)} \sqrt{\alpha_{m-1} - u}, x_m + \tfrac{1}{2} u - iAt(u) \Big) : \\ & u \in \big(u(-\epsilon), u(\epsilon)\big), \ \ x_j \in \mathbb{R}, \ \ x_1^2 + \cdots + x_a^2 - x_{a+1}^2 - \cdots - x_{m-1}^2 + 2x_m = 0 \Big\}, \end{aligned}$$

where the functions $\theta_j(u)$ and $t(u)$ are given by

$$\theta_j(u) = \begin{cases} \theta_j(0) - \dfrac{A}{2} \displaystyle\int_{u(0)}^{u} \frac{\mathrm{d}v}{(\alpha_j + v)\sqrt{Q(v) - A^2}} & j = 1, \dots, a, \\ \theta_j(0) + \dfrac{A}{2} \displaystyle\int_{u(0)}^{u} \frac{\mathrm{d}v}{(\alpha_j - v)\sqrt{Q(v) - A^2}} & j = a{+}1, \dots, m{-}1, \end{cases}$$

$$\textit{and} \qquad t(u) = \int_{u(0)}^{u} \frac{\mathrm{d}v}{2\sqrt{Q(v) - A^2}} \, .$$

This presentation has the advantage of defining N very explicitly, but the disadvantage that it is only valid for a certain range of θ, and so of t. For understanding the global properties of the solutions N, it is better to keep t as the variable, rather than u.

Next we describe the qualitative behaviour of the solutions, following the analysis of §5.4. We again divide into four cases (a)–(d).

Case (a): $A = 0$.

In this case N is an open subset of a special Lagrangian plane $\mathbb{R}^m$ in $\mathbb{C}^m$.

Case (b): $a = m - 1$ and $A > 0$.

When $m \geqslant 4$, we find that (42) and (43) admit solutions on a bounded open interval (γ, δ) with $\gamma < 0 < \delta$, such that $u(t) \to \infty$ as $t \to \gamma_+$ and $t \to \delta_-$, so that the solutions cannot be extended continuously outside (γ, δ). For $m = 3$ the solutions exist on $\mathbb{R}$, with $u(t) \to \infty$ as $t \to \pm\infty$, so we can put '$\gamma = -\infty$' and '$\delta = \infty$' in this case.

The SL m-fold N defined using the full solution interval (γ, δ) is a closed, embedded special Lagrangian m-fold diffeomorphic to $\mathbb{R}^m$, the total space of a family of paraboloids P_t in $\mathbb{C}^m$, parametrized by $t \in (\gamma, \delta)$. As $t \to \gamma_+$ and $t \to \delta_-$, these paraboloids go to infinity in $\mathbb{C}^m$, and also flatten out, so that they come to resemble hyperplanes $\mathbb{R}^{m-1}$.

At infinity, N is asymptotic (in a rather weak sense) to the union of two special Lagrangian m-planes $\mathbb{R}^m$ in $\mathbb{C}^m$ intersecting in $\big\{(0, \dots, 0, x_m) : x_m \in \mathbb{R}\big\}$, a copy of $\mathbb{R}$. We should think of these two planes as being joined when $x_m \in (-\infty, 0]$, but separated when $x_m \in (0, \infty)$.

That is, N is a kind of *connected sum* of two special Lagrangian m-planes $\mathbb{R}^m$, but a connected sum performed along an infinite interval $(-\infty, 0]$ rather

than a single point. Note that N can be regarded as a limiting case of case (b) of §5.4, in which the two special Lagrangian m-planes degenerate from meeting at a point to meeting at a line, and at the same time the x_m coordinate of their point of intersection goes to $-\infty$.

These solutions are interesting as local models for singularities of SL m-folds in Calabi–Yau m-folds. When $m = 3$ we can solve the equations very explicitly, and will do so below.

For the two remaining cases with $1 \leqslant a \leqslant m-2$ and $A > 0$, as in §5.4 we choose the constant λ uniquely such that $\alpha_j > 0$ for all j and $\sum_{j=1}^{a} \alpha_j^{-1} = \sum_{j=a+1}^{m-1} \alpha_j^{-1}$. Then $0 < A^2 \leqslant \alpha_1 \cdots \alpha_{m-1}$.

Case (c): $1 \leqslant a \leqslant m-2$ **and** $A = (\alpha_1 \cdots \alpha_{m-1})^{1/2}$**.**

As in §5.4, this is one of the SL m-folds constructed in [7, Prop. 9.3] using the 'perpendicular symmetry' idea of [7, §9], this time with $n = m-1$ and $G = \mathbb{R}$. An example of this with $a = 1$ and $m = 3$ is given in [7, Ex. 9.6].

Case (d): $1 \leqslant a \leqslant m-2$ **and** $0 < A < (\alpha_1 \cdots \alpha_{m-1})^{1/2}$**.**

As in §5.4, in this case solutions exist for all $t \in \mathbb{R}$, and u and θ are *periodic* in t, with period T. For special values of the initial data we may also arrange for $w_1, \ldots, w_{m-1}$ to be periodic with period nT for some $n \geqslant 1$.

However, by (45) we have $\operatorname{Im} \beta(t) = \operatorname{Im} \beta(0) - At$, and $A > 0$. Thus β is *never* periodic, and so the time evolution does not repeat itself. So there is no point in following the discussion of §5.5. The corresponding SL m-folds N are embedded submanifolds diffeomorphic to $\mathbb{R}^m$. For various reasons, they are not credible as local models for singularities of special Lagrangian m-folds in Calabi–Yau m-folds.

Finally, we set $m = 3$. In this case equation (42) becomes a *real linear o.d.e.* in w_1 and w_2, and so is far easier to solve. We consider the cases $a = 2$ and $a = 1$, corresponding to cases (b) and (d) above, in the next two examples.

Example 7.4 Put $m = 3$ and $a = 2$ in Theorem 7.1. Then equations (42) and (43) become

$$\frac{\mathrm{d}w_1}{\mathrm{d}t} = \bar{w}_2, \quad \frac{\mathrm{d}w_2}{\mathrm{d}t} = \bar{w}_1 \quad \text{and} \quad \frac{\mathrm{d}\beta}{\mathrm{d}t} = \overline{w_1 w_2}. \tag{46}$$

The first two equations have solutions

$$w_1 = C\mathrm{e}^t + D\mathrm{e}^{-t} \quad \text{and} \quad w_2 = \bar{C}\mathrm{e}^t - \bar{D}\mathrm{e}^{-t},$$

where $C = \frac{1}{2}\bigl(w_1(0) + \overline{w_2(0)}\bigr)$ and $D = \frac{1}{2}\bigl(w_1(0) - \overline{w_2(0)}\bigr)$. Therefore

$$\overline{w_1 w_2} = |C|^2 \mathrm{e}^{2t} - |D|^2 \mathrm{e}^{-2t} + 2i \operatorname{Im}(C\bar{D}),$$

and so integrating the third equation of (46) gives

$$\beta(t) = \tfrac{1}{2}|C|^2 \mathrm{e}^{2t} + \tfrac{1}{2}|D|^2 \mathrm{e}^{-2t} + 2i \operatorname{Im}(C\bar{D})t + E,$$

where $E=\beta(0)-\frac{1}{2}|C|^2-\frac{1}{2}|D|^2$. Thus the special Lagrangian 3-fold N in $\mathbb{C}^3$ defined in (44) is given parametrically by

$$\begin{aligned}\Bigl\{\bigl((C\mathrm{e}^t+D\mathrm{e}^{-t})x_1,(\bar C\mathrm{e}^t-\bar D\mathrm{e}^{-t})x_2,-\tfrac{1}{2}(x_1^2+x_2^2)\\ +\tfrac{1}{2}|C|^2\mathrm{e}^{2t}+\tfrac{1}{2}|D|^2\mathrm{e}^{-2t}+2i\,\mathrm{Im}(C\bar D)t+E\bigr):x_1,x_2,t\in\mathbb{R}\Bigr\}.\end{aligned}\tag{47}$$

Here we have used the equation $x_1^2+x_2^2+2x_3=0$ of (44) to eliminate x_3.

Equation (47) is a very explicit expression for a special Lagrangian 3-fold in $\mathbb{C}^3$. Case (a) above, with $A=0$, corresponds to $\mathrm{Im}(C\bar D)=0$, and in this case N is a subset of an affine special Lagrangian 3-plane $\mathbb{R}^3$ in $\mathbb{C}^3$. If $\mathrm{Im}(C\bar D)\neq 0$ then N is an embedded submanifold diffeomorphic to $\mathbb{R}^3$, with coordinates (x_1,x_2,t).

Example 7.5 Put $m=3$ and $a=1$ in Theorem 7.1. Then equations (42) and (43) become

$$\frac{\mathrm{d}w_1}{\mathrm{d}t}=\bar w_2,\quad \frac{\mathrm{d}w_2}{\mathrm{d}t}=-\bar w_1\quad\text{and}\quad\frac{\mathrm{d}\beta}{\mathrm{d}t}=\overline{w_1w_2}.\tag{48}$$

The first two equations have solutions

$$w_1=C\mathrm{e}^{it}+D\mathrm{e}^{-it}\quad\text{and}\quad w_2=i\bar D\mathrm{e}^{it}-i\bar C\mathrm{e}^{-it},$$

where $C=\frac{1}{2}\bigl(w_1(0)-i\overline{w_2(0)}\bigr)$ and $D=\frac{1}{2}\bigl(w_1(0)+i\overline{w_2(0)}\bigr)$. Therefore

$$\overline{w_1w_2}=iC\bar D\mathrm{e}^{2it}-i\bar CD\mathrm{e}^{-2it}+i\bigl(|C|^2-|D|^2\bigr),$$

and so integrating the third equation of (48) gives

$$\beta(t)=\tfrac{1}{2}C\bar D\mathrm{e}^{2it}+\tfrac{1}{2}\bar CD\mathrm{e}^{-2it}+i\bigl(|C|^2-|D|^2\bigr)t+E,$$

where $E=\beta(0)-\mathrm{Re}(\bar CD)$. Thus the SL 3-fold N in $\mathbb{C}^3$ defined in (44) is given parametrically by

$$\begin{aligned}\Bigl\{\bigl((C\mathrm{e}^{it}+D\mathrm{e}^{-it})x_1,(i\bar D\mathrm{e}^{it}-i\bar C\mathrm{e}^{-it})x_2,\tfrac{1}{2}(x_2^2-x_1^2)\\ +\tfrac{1}{2}C\bar D\mathrm{e}^{2it}+\tfrac{1}{2}\bar CD\mathrm{e}^{-2it}+i\bigl(|C|^2-|D|^2\bigr)t+E\bigr):x_1,x_2,t\in\mathbb{R}\Bigr\}.\end{aligned}\tag{49}$$

Here we have used the equation $x_1^2-x_2^2+2x_3=0$ of (44) to eliminate x_3.

Case (a) above, with $A=0$, corresponds to $|C|=|D|$, and in this case N is a subset of an affine special Lagrangian 3-plane $\mathbb{R}^3$ in $\mathbb{C}^3$. If $|C|\neq|D|$ then N is an embedded submanifold diffeomorphic to $\mathbb{R}^3$, with coordinates (x_1,x_2,t). The two cases $C=0$ and $D=0$ are constructed by [7, Prop. 9.3] with $n=2$, $m=3$ and $G=\mathbb{R}$, as in [7, Ex. 9.6] and case (c) above, with the symmetry group G of N acting by $(x_1,x_2,t)\mapsto(x_1,x_2,t+c)$.

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